Nonlinear friction effect in air-pressure forcing of the flow through the Strait of Gibraltar

Nonlinear friction
Gibraltar
Barotropic flow

Frottement non linéaire
Gibraltar
Flux barotrope

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ABSTRACT

The simple analytical two-strait and two-basin model built by (Candela *et al.*, 1989), which well describes the barotropic flow through the Strait of Gibraltar forced by homogeneous fields of oscillating atmospheric pressure over the Eastern and Western Mediterranean, was improved by allowing the coefficient of linear bottom friction λ to be frequency-dependent. The coefficient λ is here proportional to the magnitude of the volume flow through the Strait of Gibraltar, and therefore increases with the frequency at very low frequencies, starting from zero. After reaching the extreme, which was tuned to be close to the value of frequency-independent λ, the coefficient slowly decreases with frequency. Both approaches, with frequency-dependent and independent λ give almost the same frequency dependence for all variables, therefore the disagreement between model values and cross-spectral analysis remains at high frequencies. Using the values of cross-section dimensions proposed by Candela *et al.* the estimates of the drag coefficient are by about two orders of magnitude too high in both approaches. If the geostrophic control of the flow through the Strait of Gibraltar plays a role at low frequency range also when friction is included, then its upper frequency limit is estimated to be ~ 4×10^{-3} cph.

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RÉSUMÉ

Frottement non linéaire dû au forçage de la pression de l'air sur le flux dans le détroit de Gibraltar

Le modèle analytique simple du détroit de Gibraltar proposé par Candela *et al.* (1989), avec deux détroits et deux bassins, décrit correctement le flux barotrope forcé par des champs homogènes de pression atmosphérique qui oscillent entre l'est et l'ouest de la Méditerranée. Ce modèle est amélioré lorsque le coefficient linéaire de frottement sur le fond varie avec la fréquence. Ici le coefficient est proportionnel au flux traversant le détroit de Gibraltar ; il augmente donc avec la fréquence aux très bassses fréquences, en partant de zéro. Après avoir atteint un maximum proche de la valeur indépendante de la fréquence, le coefficient décroît légèrement lorsque la fréquence continue à augmenter. Les deux approches, avec coefficient variable ou indépendant de la fréquence, donnent pour tous les autres paramètres la même loi de variation avec la fréquence ; par conséquent, la différence entre les valeurs du modèle et l'analyse spectrale persiste aux fréquences élevées. Les valeurs des dimensions de la section proposés par Candela *et al.* conduisent à des estimations du coefficient de frottement qui sont trop élevées d'environ deux ordres de grandeur dans les deux approches. Si le contrôle géostrophique du flux à travers le détroit de Gibraltar joue un rôle dans la gamme des basses fréquences, même lorsque le frottement est pris en compte, la limite supérieure de la fréquence est estimée de l'ordre de 4×10^{-3} cycles par heure.

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INTRODUCTION

Nonlinear friction has mainly been invoked in the tidal analysis of shallow seas. In tidal spectroscopy in 1D channels closed at one end, odd harmonics due to bottom friction were evaluated approximately by Gallagher and Munk (1971) and a perturbation method was applied for them by Kabbaj and Le Provost (1980). They also compared both analytical approximations with numerical solutions of equations of motion. Iterative procedure, the renormalization, was used by Zimmerman (1982) for a quadratically-damped forced oscillator, and for a co-oscillated basin connected with an open sea through a narrow strait with nonlinear friction, which also behaves like an oscillator (Zimmerman, 1991; 1992).

All the above-mentioned methods concerning nonlinear friction were applied to problems, which reduce to an oscillator of some kind. Here, an attempt has been made to make the nonlinear correction of the model built by Candela, Winant and Bryden (1989), hereafter referred as CWB, which seems to the author to be one of the most accomplished of the "two-strait, two-basin" series of models, developed for describing the barotropic subinertial flows through straits of the Mediterranean Sea forced by an oscillated atmospheric pressure field. From the linearization of the bottom friction stress it follows that the coefficient of the linearized friction stress is proportional to the amplitude of velocity of the rectilinear flow. Therefore the coefficient involved in linearized friction stress is frequency-dependent. This last relaxation of the coefficient out from the constant value was also suggested by CWB in their conclusions as a possible next step in the parametrization of the model.

For the rectilinear flow, quadratic bottom friction stress produces higher (odd) harmonics. From the Fourier analysis of bottom friction stress it follows that only the first six of them are reasonably significant. We shall require that the bottom stress of the rectilinear motion should be correctly approximated for the fundamental harmonic. This means also that the average energy loss within one cycle of rectilinear motion is the same in both approaches, with and without linearization (Bowden, 1983). This goal is also achieved with a linear approximation of the first term in the Fourier series of the bottom friction stress for the rectilinear flow. The well-known relation between friction parameters (Zimmerman, 1991) is then obtained

$$\lambda = \frac{8C_dU}{3\pi H} \tag{1}$$

where C_d is the drag coefficient and U is the amplitude of the vertically averaged velocity of the rectilinear flow of the water column of depth H.

MODEL PRESENTATION

The simple model developed by CWB satisfactory describes the barotropic volume flow through the Strait of Gibraltar, where the forcing agent is the first mode of EOF analysis of the atmospheric pressure field over the Mediterranean Sea. We shall adopt their model in our consideration of nonlinear friction. The conservations of volume in the first (W Mediterranean) and the second (E Mediterranean) basin are therefore given as

$$A_1 \frac{\partial \eta_1}{\partial_t} = q_G - q_S \tag{2.1}$$

$$A_2 \frac{\partial \eta_2}{\partial_t} = q_S \tag{2.2}$$

where $q_G = u_1A_G$, $q_S = u_2A_S$ are the volume flows through the first (Gibraltar) strait, and the second (Sicily) strait, A_G and A_S are their cross-section areas, while η_1 and A_1 (η_2 and A_2) are the uniform sea-level elevation and the surface area of the first (second) basin. The subsurface uniform pressure in each basin is written simply as

$$p_1 = p_{a1} + \rho g \eta_1 \tag{2.3}$$

$$p_2 = p_{a2} + \rho g \eta_2 \tag{2.4}$$

where p_{a1} and p_{a2} are the spatially averaged atmospheric pressures over basin 1 and 2, respectively. The last two equations are the equations of motion through straits

$$\frac{\partial q_G}{\partial_t} = \frac{-A_G p_1}{\rho L_G} - \lambda q_G \tag{2.5}$$

$$\frac{\partial q_S}{\partial_t} = \frac{-A_S(p_2 - p_1)}{\rho L_S} \tag{2.6}$$

In the equation of motion through the Strait of Sicily the pressure gradient was approximated with $(p_2 - p_1)/Ls$, where Ls is the length of the strait. For this strait, following the same argument of CWB of six times larger cross-sectional area of Sicily than of Gibraltar, the water velocities and frictional effects should be small, therefore the frictional term was omitted as in the CWB model. Further clarification of this approximation will follow in the discussion. The pressure gradient along the Strait of Gibraltar was approximated with p_1/L_G, where L_G is the length of the Strait of Gibraltar, meaning that isostatic pressure variations due to the atmosphere are ignored on the Atlantic side of Gibraltar. All dimensions of straits and basins used by CWB will also be used here. CWB analyzed observations of atmospheric pressure sampled every six hours from 64 coastal meteorological stations using the EOF method. They revealed that 65 % of the variance accounts for the spatial structure of the first mode, which represents a homogeneous, standing pattern without sign change. In our case the only forcing agent at subinertial frequencies are the atmospheric pressure fields averaged over each of the basins. So we shall write them as

$$p_{a1} = \mathcal{R}\{P_{a1}\, e^{i\omega t}\}$$

$$p_{a2} = \mathcal{R}\{P_{a2}\, e^{i\omega t}\} \tag{2.7}$$

We shall again look for the solutions in the form of trigonometric functions

$$\eta_{1,2} = \mathcal{R}\{Z_{1,2}\, e^{i\omega t}\}$$

$$p_{1,2} = \mathcal{R}\{P_{1,2}\, e^{i\omega t}\}$$

$$q_{G,S} = \mathcal{R}\{Q_{G,S}\, e^{i\omega t}\} \tag{2.8}$$

Inserting them in (2.1) to (2.6), together with forcing functions (2.7), for which amplitudes P_{a1} and P_{a2} are real, we obtain coupled equations for the complex amplitudes

$$i\omega A_1 Z_1 = Q_G - Q \quad (2.9)$$

$$i\omega A_2 Z_2 = Q_S \quad (2.10)$$

$$P_1 = P_{a1} + \rho g Z_1 \quad (2.11)$$

$$P_2 = P_{a2} + \rho g Z_2 \quad (2.12)$$

$$i\omega Q_G = -\frac{A_G P_1}{\rho L_G} - \lambda Q_G \quad (2.13)$$

$$i\omega Q_S = -\frac{A_S (P_2 - P_1)}{\rho L_S} \quad (2.14)$$

Only the real parts of all quantities written in a complex form are considered at the end of the derivation procedure. The above system of amplitude equations is linear if the coefficient λ is not a function of the flow through Gibraltar Strait, and was estimated by CWB from $\lambda = C_d U_o/H_G$ to be $1.27 \times 10^{-5} s^{-1}$ where for the hydraulic depth of the Strait of Gibraltar H_G of 120 m was used, the drag coefficient $C_d = 3 \times 10^{-3}$ and the amplitude of velocity $U_o = 0.5$ m/s. But, as will be pointed out later, four times larger value had to be used to adjust the model with the data of the flow through the Strait of Gibraltar. Here, according to (1.1), λ will be proportional to the amplitude of the volume flow Q_G, but since this is shifted in phase with regard to the atmospheric pressure fields, the modulus $|Q_G|$ should be used

$$\lambda = \frac{8C_d}{3\pi A_G H_G} |Q_G| \equiv C_D |Q_G| \quad (2.15)$$

where $C_D \approx 7.2 \times 10^{-12}\ m^{-3}$ if we use $A_G = 2.95 \times 10^6\ m^2$, $H_G = 120$ m, and $C_d = 3 \times 10^{-3}$. Later, different values of C_D will be considered to obtain model results similar to results of CWB. Inserting Z_1 from (2.9) and P_1 from (2.11) into (2.13) we obtain the first relation between Q_G and Q_S

$$Q_G \left[1 - \frac{gA_G}{L_G A_1 \omega^2} - \frac{i\lambda}{\omega}\right] = -\frac{gA_G Q_s}{A_1 \omega^2 L_G} + \frac{iA_G P_{a1}}{\rho\omega L_G} \quad (2.16)$$

For the second relation we put Z2 from (2.12), P2 from (2.14), and P1 from (2.13) into (2.10)

$$Q_S \left[1 - \frac{\omega^2 L_s A_2}{gA_s}\right] = -\frac{L_G \omega A_2 Q_G [\omega - i\lambda]}{gA_G} + \frac{i\omega A_2 P_{a2}}{\rho g} \quad (2.17)$$

Before solving the system of (2.16) and (2.17) it seems to be more appropriate to introduce another two constants

$$\Omega_I^2 = \frac{gA_G}{L_G A_1};\ \Omega_{II}^2 = \frac{gA_s}{L_s A_2}$$

For values of $L_G = 6 \times 10^4$ m, $L_S = 10^5$ m, $A_G = 2.951 \times 10^6\ m^2$, $A_S = 1.8926 \times 10^7\ m^2$, $A_1 = 8.646 \times 10^{11}\ m^2$, and $A_2 = 1.6703 \times 10^{12}\ m^2$ (Tab. 2 of CWB) they become: $\Omega_I^2 = 0.558 \times 10^{-9}\ s^{-2}$ and $\Omega_{II}^2 = 1.11 \times 10^{-9}\ s^{-2}$. Considering them as angular frequencies, they correspond to the periods of ~ 2.2 days and ~ 3.1 days. We shall express the amplitudes of the atmospheric pressure fields in elevation units $Z_{a1} = P_{a1}/(\rho g)$ and $Z_{a2} = P_{a2}/(\rho g)$, where ρ is the density of the sea water. The resulting equation for Q_G may be written in a similar form to that obtained by CWB

$$Q_G [\mathcal{B} + i\lambda \mathcal{C}] = i\omega \left[A_1 Z_{a1} \left(1 - \frac{\omega^2}{\Omega_{II}^2}\right) + A_2 Z_{a2}\right] \quad (2.18)$$

with

$$\mathcal{B} = -1 + \omega^2 \left[\frac{1}{\Omega_I^2}\left(1 + \frac{A_2}{A_1}\right) + \frac{1}{\Omega_{II}^2}\right] - \frac{\omega^4}{\Omega_I^2 \Omega_{II}^2}$$

$$\mathcal{C} = \frac{\omega}{\Omega_I^2}\left[\frac{\omega^2}{\Omega_{II}^2} - 1 - \frac{A_2}{A_1}\right]$$

where D_0 in CWB was replaced with $\mathcal{B}$. Now, we shall multiply both sides of the above complex equation with their conjugate expressions, and we shall introduce λ from (2.15). The amplitude equation for $|Q_G|$ finally follows as

$$|Q_G|^4 (\mathcal{C_D C})^2 + |Q_G|^2 \mathcal{B}^2 - \mathcal{A}^2 = 0 \quad (2.19)$$

where coefficient $\mathcal{A}$ is

$$\mathcal{A} = A_1 Z_{a1} \left(1 - \frac{\omega^2}{\Omega_{II}^2}\right) + A_2 Z_{a2}.$$

Coefficient $\mathcal{C} = 0$ when $\omega = \Omega_{II}(1 + A_2/A_1)^{1/2} \approx 5.7 \times 10^{-5}\ s^{-1}$ and the corresponding period is around 30.6 hours. We shall denote this particular angular frequency with ω_0. The solution for $|Q_G|$ for this angular frequency does not depend on the friction coefficient $\mathcal{C_D}$

$|Q_G| = \omega_0 A_1 |Z_{a2} - Z_{a1}|$.

For $C \neq 0$ the single positive solution of (2.19) is easily obtained as

$$|Q_G| = \frac{\sqrt{-\mathcal{B}^2 + \sqrt{\mathcal{B}^4 + 4(\mathcal{ACC_D})^2}}}{\sqrt{2}\ \mathcal{C_D}|\mathcal{C}|} \quad (2.20)$$

In the above expression the volume flow through the Strait of Gibraltar is a more complicated function of angular frequency than $|Q_G|$ obtained by CWB for λ independent of ω, which follows directly from (2.18)

$$|Q_G| = \frac{\omega\ |\mathcal{A}|}{\sqrt{\mathcal{B}^2 + (\lambda\mathcal{C})^2}} \quad (2.21)$$

Multiplication of (2.20) by a constant $\mathcal{C_D}$ gives us the dependence of λ on ω, according to (2.15), and also on the forcing amplitudes Z_{a1} and Z_{a2}, which is a characteristic of nonlinear friction law.

DISCUSSION-CONCLUSION

The derivation of all unknowns for λ being a function of frequency, may start with Q_G from (2.18), in which we insert $\lambda = \mathcal{C_D}|Q_G|$, with $|Q_G|$ from (2.20). Plots of their abso-

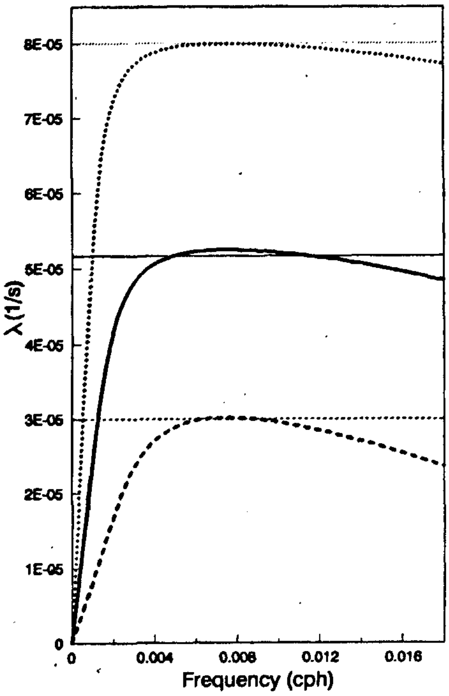

Figure 1

Coefficient of friction λ as a function of frequency for $C_D = 5.6 \times 10^{-10}$ m^{-3} (solid line), $C_D = 1.85 \times 10^{-10}$ m^{-3}, and $C_D = 13.0 \times 10^{-10}$ m^{-3}. Thin horizontal lines represent constant values of λ used by CWB (see text).

lute values, and phase shifts with regard to the oscillations of the atmospheric pressure as a function of frequency will be presented here. Atmospheric pressure amplitudes will be normalized $P_{a1} = P_{a2} = 1$ mbar since we intend to compare the model results with the cross-spectral analysis. In Figure 1 the coefficient of friction is plotted against frequency for three values of parameter C_D. These have been chosen on the criteria of satisfactory agreement between the magnitudes (gains) $|Q_G|$ calculated for frequency-dependent λ and those calculated for frequency-independent λ (Fig. 2). It turned out that the satisfactory agreement was achieved for C_D of 5.6×10^{-10} m^{-3}, 1.85×10^{-10} m^{-3}, and 13.0×10^{-10} m^{-3}.

Values of frequency-independent λ used by CWB in the model are 5.17×10^{-5} s^{-1}, 3.0×10^{-5} s^{-1}, and 8.0×10^{-5} s^{-1}. They have been also used here. The coefficient of linear friction increases with frequency (together with $|Q_G|$) almost linearly at very low frequencies, with a smaller rate for large values of C_D. After reaching the extreme, which is very close to the adequate value of λ used by CWB, the coefficient slowly decreases with frequency, this time with a higher rate for large values of C_D. Bearing in mind the value for $A_G H_G \approx 3.54 \times 10^8$ m^3, used by CWB, this implies that values for the drag coefficient C_d are to be 0.198, 0.065, and 0.462, respectively, which is certainly by about two orders of magnitude too high.

Let the frequency-dependent λ be equal to the frequency-independent value, say $\lambda = 5.17 \times 10^{-5}$ s^{-1}. From Figure 1 we can easily obtain the two frequencies at which this happens as 0.0048 and 0.0113 cph. Also at these frequencies, gains $|Q_G|$ (Fig. 2) of both models exactly meet and are equal (solid lines) to 0.092 Sv (1 Sv $\equiv 10^6$ m^3 s^{-1}). Using this value for the volume flow in (2.15) $C_d \approx 0.23$ which is again by two orders of magnitude too high. A pronounced reduction of the "hydraulic" depth H_G by about one order of magnitude would give us reasonable values for the drag coefficient for both models using different concepts of λ.

Nonetheless, both approaches give quite similar behaviour for the volume flow through the Strait of Gibraltar, and they both fail in describing the flow at frequencies higher than 8×10^{-3} cph. At these frequencies the barotropic flow at Gibraltar, according to EOF analysis of CWB, is not related to the first EOF mode of atmospheric pressure, supposed to be in forcing function, over the Mediterranean. The coherence between atmospheric pressure and the flow through Gibraltar is small in this frequency range. The values of the cross-spectral analysis of the volume flow through the Strait of Gibraltar and air pressure were digitized from figures of CWB and may therefore be considered as indicative only. Careful comparison of our model results for $|Q_G|$ with frequency-independent λ, with values calculated by CWB show that our model values are about 9 % higher.

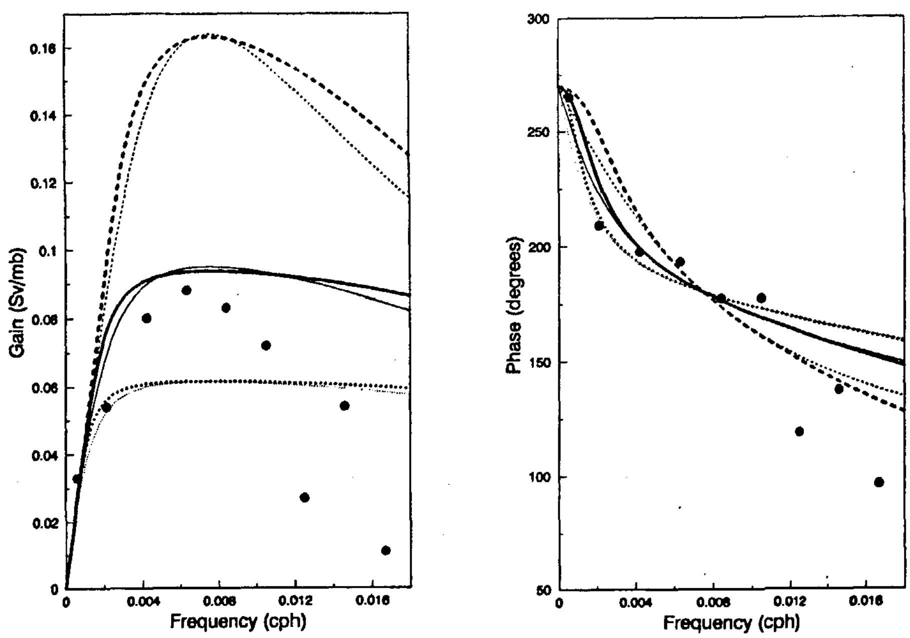

Figure 2

Magnitudes and phase shifts of the volume flow Q_G for λ being a function of frequency (thick lines), and λ being a constant (thin lines). The first group of results are drawn for three values of constant C_D: 5.6×10^{-10} m^{-3} (solid line), 1.85×10^{-10} m^{-3} (dashed line), and 13.0×10^{-10} m^{-3} (dotted line). For the second group of results plots are for the values of constant λ: 5.17×10^{-5} s^{-1}, 3.0×10^{-5} s^{-1}, and 8.0×10^{-5} s^{-1}. Full circles are the data of cross-spectral analysis performed by CWB.

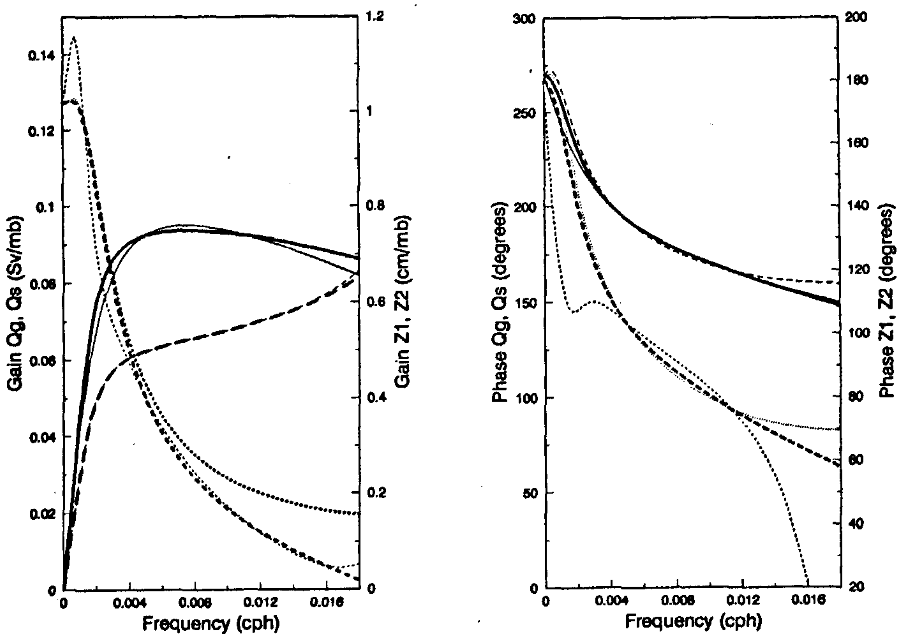

Figure 3

Magnitudes and phase shifts of the volume flows Q_G (full lines), Q_S (long dashes) and sea-level elevations Z_1 (dashed lines), and Z_2 (dotted lines) for λ being a function of frequency (thick lines), where C_D: = 5.6×10^{-10} m^{-3}, and for $\lambda = 5.17 \times 10^{-5}$ s^{-1} (thin lines). Lines of phase shifts between the atmospheric pressure signal and both sea-level elevations for λ varying with frequency (thick dashed and thick dotted line) are one above the other.

The reason for this discrepancy is unknown to the author. Still, the shape of variations of $|Q_G|$ certainly follows the same curve, and we are again faced in this simple model with almost the same response as in CWB. An introduction of the variations of friction coefficient with frequency, which is based on the use of quadratic friction in the basic harmonic, obviously does not lead to a better agreement of the model with already mentioned cross-spectral analysis. Unfortunately, the time series of the relevant data used for the cross-spectral analysis were not long enough (thirteen months of data) to provide the frequency resolution necessary for checking abrupt variations in λ at low frequency range. Magnitudes of the volume flow through the Strait of Sicily (Q_S; Fig. 3), and of average elevation of the Western (Z_1) and Eastern (Z_2) Mediterranean, follow curves which are almost covered for both cases, with λ being frequency-dependent, or independent. There are some differences in magnitude of Z_1. At frequencies around 7×10^{-4} cph $|Z_1|$ overshoots the response of inverted barometer by about 16 %

flow through the Strait of Gibraltar. The arguments of CWB for dropping the friction term through the Strait of Sicily, based on the geometry ($A_S > A_G$) of this strait, which is much wider than the Strait of Gibraltar, are not too convincing (it seems that wider Sicily strait is also quite shallow, which makes the friction important). Involvement of λ_S, which should also be proportional to the magnitude of the volume flow Q_S, seriously complicates the system of equations, and terms like $\lambda_G\lambda_S$ appear on the way to the amplitude equations for both volume flows. But although the uniqueness of the solutions should be verified, they would perhaps be in a better agreement with measurements at frequencies above 8×10^{-3} cph. This should perhaps be one of the improvements in the model for future work.

Another possibility of improving the model would be the introduction of the Fourier series of the bottom friction stress with higher odd harmonics. But, since the energy lost within one cycle is already well parameterized with the are needed to examine the behaviour of the straits at low frequency range. An attempt might be made to combine both mechanisms, the geostrophic control and the linearization of friction which provides the correct energy dissipation within one cycle, by simply allowing λ to be the sum of a constant part according to the geostrophic control, together with a frequency-dependent part, which is proportional to the volume flow through the strait. This would actually be tantamount to considering λ to be a "generic form of constraint" (CWB). The amplitude equation similar to (2.19) could probably be solved. But, in addition to other serious limitations in the use of geostrophic control (*i. e.* finite basins in which coastal Kelvin waves may propagate; Wright, 1987), the physical grounds for doing this simple addition do not appear to be strong.

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