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A bottom stress formulation for storm surge problems

Bottom stress
Storm surge
Three-dimensional modeling
Contrainte sur le fond
Onde de tempête
Modélisation à trois dimensions

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ABSTRACT

In view of the importance of any kind of numerical modelling in shallow seas a new bottom stress formulation based on non-steady Ekman theory is derived for the homogeneous sea.
This is achieved by finite differencing in horizontal space and time and by analytical integration in the vertical. The bottom stress becomes dependent on wind stress, sealevel slope and the time history of the current. The actual computations need only one previous time level. Different bottom stress formulations including the *a priori* ones are applied for a model basin under the simplified conditions of a sudden imposed constant or periodic wind. The new formulation produces a different phase behaviour, a smaller damping, a higher maximum set up and a higher steady state set up for the sealevel. The only free parameter for the derived stress formulation is eddy viscosity which is determined on the basis of steady-state theory and of an empirical relation between wind speed and surface current.

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RÉSUMÉ

Formulation de la contrainte des ondes de tempête sur le fond

Compte tenu de l'importance de tout ce qui concerne la modélisation numérique en mer peu profonde, une nouvelle formulation de la contrainte sur le fond, basée sur une théorie d'Ekman non permanente est établie pour la mer homogène.
Ceci est obtenu par une différenciation finie dans l'espace horizontal et dans le temps, et par une intégration analytique sur la verticale. La contrainte sur le fond dépend alors de la force du vent, de la pente de la surface marine et de l'évolution antérieure du courant. En pratique, le calcul ne s'appuie que sur le pas de temps précédent.
Différentes formulations de la contrainte sur le fond, en particulier celles *a priori*, sont associées à un modèle de bassin dans les hypothèses simplifiées d'un vent constant ou périodique appliqué de façon soudaine. La nouvelle formulation présente un comportement de phase différent, un amortissement plus faible, une élévation maximale et une élévation constante plus fortes du niveau de la mer. Le seul paramètre libre pour la formulation de la contrainte sur le fond est la viscosité turbulente qui est déterminée à partir de la théorie d'état permanent et d'une relation empirique entre la vitesse du vent et le courant de surface.

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INTRODUCTION

The choice of bottom friction formulation is essential for all kinds of marine circulation models especially for those which deal with the shallower parts of the oceans, for shelf areas, for estuaries, fjords and river mouths.

On one hand, these areas provide main sites for industrial development, on the other hand they are natural nursing grounds with rich nutrients for marine plants and animals.

In order to study the sensitive chemical and biological balance of these areas sophisticated ecosystem models are used which need the fundamental base of satisfactory circulation models. For many applications the quality of physical models, however, is based on the bottom stress formulation.

As has been shown by the numerical computations of storm surges (Hansen, 1956; Platzman, 1963; Jelesnianski, 1967), Ekman type equations based on a constant eddy viscosity coefficient can be used successfully to predict short-time phenomena of sealevel variations. As Platzman (1963) or Jelesnianski (1967) we will consider the application of the Ekman (1905) solution of the current problem in order to derive a formulation of the stress at the bottom. This stress, afterwards, can be introduced into the mass transport equations to predict storm surges. When dealing with the formulation of bottom stress, generally speaking, two approaches are feasible: on the one hand, the bottom stress can be chosen *a priori*, say from hydraulic's experiments (Hansen, 1956), on the other hand, the stress can be taken from the Ekman equations and then it is expressed as a function of sea slope and wind stress.

The second way was clearly stated by Welander (1957) who showed that in the time-dependent motion the local velocity profile and the bottom stress are uniquely determined by the local time history of the wind stress and the surface slope.

Platzman (1963) formulated the bottom stress through the Ekman equations and obtained a differential operator expression. A series expansion of this operator provides an insight into the problem. Further approximations lead to a quasi-linear friction formulation which is used for comparison in this paper [equation (35)]. Because of inherent convergence problems in the differential operator's series expansion, Jelesnianski (1970) derived a new formulation based on integral operators starting from Welander's (1957) and Platzman's (1963) experiences. The bottom stress is formulated as a sum of convolution integrals in time: of surface slope and of wind stress. The integral kernels are infinite series of exponential functions. The method proved itself as a useful tool, though it needs complicated recursion formulas for the kernels of the convolution integrals.

Our aim is to formulate the problem through the difference-differential equations, and to solve one part of the problem analytically, i. e. the integration over the depth, and the other part, i. e. the time stepping, numerically. This allows also the prediction of current profiles in parallel with mass transport and sealevel.

The modeling of three-dimensional currents through numerical integration of a two-dimensional problem plus analytical integration of the vertical dependence can be found in the papers of Forristall (1974), Forristall *et al.* (1977) or Nihoul (1977). Forristall, in both papers, is mainly following the lines of Jelesnianski (1970) changing from a no-slip condition (1974) to a slip condition at the bottom (1977). He claimes to arrive at a better agreement between model and observations by using a slip condition. However, because an additional constant, the slip parameter, can be used to adjust the model, the better agreement is not quite surprising. Nihoul (1977), solving the same problem analytically for no-slip at the bottom and vertically variable eddy diffusivity, is mainly interested in the three-dimensional current profile rather than in the time history of the bottom stress. In this paper, eddy viscosity is constant, but cannot be chosen arbitrarily to tune the model. It is chosen according to a modification of a method proposed by Felzenbaum (1960).

Different bottom stress formulations together with the one presented are applied to compute the time history of the sealevel in a model basin.

EQUATIONS

We shall use the following set of equations :

$$\frac{\partial u}{\partial t}=g\frac{\partial \zeta}{\partial x}+fv+\mu\frac{\partial^2 u}{\partial z^2}, \qquad (1)$$

$$\frac{\partial v}{\partial t}=g\frac{\partial \zeta}{\partial y}-fu+\mu\frac{\partial^2 v}{\partial z^2}, \qquad (2)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad (3)$$

As usual, u, v, w are the components of velocity related to the x, y, z axes, g is the earth's gravity acceleration, f the Coriolis parameter, μ the eddy viscosity coefficient, ζ the sealevel variation positive in direction of z. The z-axis is pointing toward the earth's centre, x to the East and y to the North, i. e. x, y, z form a left handed system. A pertinent set of boundary conditions for (1)-(3) is

$$-\mu\frac{\partial u}{\partial z}=\tau_x; \qquad -\mu\frac{\partial v}{\partial z}=\tau_y \qquad \text{at} \quad z=0, \qquad (4)$$

and

$$u=v=w=0, \qquad \text{at} \quad z=H(x, y), \qquad (5)$$

where τ_x, τ_y are the components of surface stress and H the water depth.

To describe the sealevel in the equations (1) and (2) the system of vertically integrated equations of continuity and motion, will be applied

$$\frac{\partial \zeta}{\partial t}=\frac{\partial M_x}{\partial x}+\frac{\partial M_y}{\partial y}, \tag{6}$$

$$\frac{\partial M_x}{\partial t}=g\,H\frac{\partial \zeta}{\partial x}+fM_y+(\tau_x-\tau_{b,x}), \tag{7}$$

$$\frac{\partial M_y}{\partial t}=g\,H\frac{\partial \zeta}{\partial y}-fM_x+(\tau_y-\tau_{b,y}), \tag{8}$$

where M_x, M_y are the volume transport components and $\tau_{b,x}$, $\tau_{b,y}$ denote the bottom stress components defined by

$$\tau_{b,x}=-\mu\frac{\partial u}{\partial z}\bigg|_{z=H}; \qquad \tau_{b,y}=-\mu\frac{\partial v}{\partial z}\bigg|_{z=H}. \tag{9}$$

The lateral boundary condition for the above system follows from the fact that the volume transport component normal to the coast has to vanish. In the following, the slightly incorrect term "mass" transport is used sometimes instead of "volume" transport.

STEADY EKMAN PROBLEM

Let us set $\partial u/\partial t=0$; $\partial v/\partial t=0$, in equations (1) and (2), next multiply the second equation by the imaginary number i and afterwards add them side by side. Finally, we arrive at the equation

$$\frac{d^2 s}{dz^2}-\alpha^2 s=Q, \tag{10}$$

where

$$s=u+iv, \qquad \alpha^2=\frac{fi}{\mu}, \qquad Q=-\frac{g}{\mu}\frac{\partial \zeta}{\partial x}-\frac{g}{\mu}i\frac{\partial \zeta}{\partial y}.$$

The general solution to equation (10) with the boundary conditions (4) and (5) takes the form

$$s=\frac{\tau}{\mu\alpha}\frac{\operatorname{sh}\alpha(H-z)}{\operatorname{ch}\alpha H}+I_z-I_H\frac{\operatorname{ch}\alpha z}{\operatorname{ch}\alpha H}, \tag{11}$$

where

$$\tau=\tau_x+i\,\tau_y, \tag{12}$$

$$I_z=\frac{1}{\alpha}\int_0^z Q(\eta)\operatorname{sh}\alpha(z-\eta)\,d\eta$$

and

$$I_H=I_z\,|_{z=H}. \tag{13}$$

The general solution (11) will be of use later on, but now, since Q is constant with respect to z, the solution of the Ekman problem simplifies to

$$s=\frac{\tau}{\mu\alpha}\frac{\operatorname{sh}\alpha(H-z)}{\operatorname{ch}\alpha H}+\frac{Q}{\alpha^2}\left(\frac{\operatorname{ch}\alpha z}{\operatorname{ch}\alpha H}-1\right). \tag{11 a}$$

Due to the lack of information on Q, an additional equation has to be specified, usually it is the equation of volume transport (Welander, 1957). To derive this equation we resort to the equation of continuity (6) which in the steady case

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_y}{\partial y}=0, \tag{14}$$

allows to introduce the stream function ψ instead of the volume transport components

$$M_x=\frac{\partial \psi}{\partial y}; \qquad M_y=-\frac{\partial \psi}{\partial x}. \tag{15}$$

There are two ways of formulating the volume transport equation. First the system (1) and (2) may be integrated from the surface to the bottom (with $\partial/\partial t=0$), in the way the equations (7) and (8) are obtained, on the other hand the vertical distribution of current (11) can also be integrated. An interesting question arises: Do both equations possess a similar form? The answer is —certainly not. Integration of equations (1) and (2) introduces stresses at the bottom and at the surface, but in (11 *a*) the condition at the bottom states that the velocity should decay there. Due to this, a unique possibility arises; the bottom stress as derived from the Ekman solution (11 *a*) can be inserted into the mass transport equations.

BOTTOM STRESS FORMULATION-STEADY CASE

To solve this problem the equations (7) and (8) are taken for the steady state, and the stress is defined from (11 *a*) as

$$\mu\frac{ds}{dz}\bigg|_{z=H}=-\frac{\tau}{\operatorname{ch}\alpha H}+\frac{\mu Q}{\alpha}\frac{\operatorname{sh}\alpha H}{\operatorname{ch}\alpha H}, \tag{16}$$

or writing the real and imaginary parts explicitly,

$$\mu\frac{\partial u}{\partial z}\bigg|_{z=H}=-\frac{g}{4\alpha A}\left(\frac{\partial \zeta}{\partial x}B+C\frac{\partial \zeta}{\partial y}\right)-\frac{1}{A}(D\,\tau_x+E\,\tau_y), \tag{17}$$

$$\mu\frac{\partial v}{\partial z}\bigg|_{z=H}=-\frac{g}{4\alpha A}\left(-C\frac{\partial \zeta}{\partial x}+B\frac{\partial \zeta}{\partial y}\right)-\frac{1}{A}(-E\,\tau_x+D\,\tau_y). \tag{18}$$

In (17) and (18) the following notation is used

$$A=\operatorname{ch}^2\alpha_1 H-\sin^2\alpha_1 H;$$
$$B=\sin 2\alpha_1 H+\operatorname{sh}2\alpha_1 H,$$
$$C=\operatorname{sh}2\alpha_1 H-\sin 2\alpha_1 H;$$
$$D=\operatorname{ch}\alpha_1 H\cos\alpha_1 H,$$
$$E=\sin\alpha_1 H\operatorname{sh}\alpha_1 H.$$

$$\alpha_1=\sqrt{\frac{f}{2\mu}}. \tag{19}$$

If the sealevel slope components are known in (17) and (18) one may easily insert the bottom stress into the equations of mass transport. A usual procedure to obtain the slope is to derive the equation of mass transport through vertical integration of equation (11 *a*) and to apply the stream function notation. The elliptical problem for the stream function can be stated and solved, see e. g. Welander (1957), Felzenbaum (1960). We do not intend to pursue this way, since we are interested in the nonsteady formulation. Let us instead use Kazakov's (1976) proposition and insert the stress, which follows from the steady Ekman solution (17), (18), into the nonsteady set of mass transport equations (7) and (8). Thus

$$\frac{\partial M_x}{\partial t}=g\,H\frac{\partial \zeta}{\partial x}+f\,M_y+\tau_x - \frac{g}{4\alpha A}\left(B\frac{\partial \zeta}{\partial x}+C\frac{\partial \zeta}{\partial y}\right)-\frac{1}{A}(D\tau_x+E\tau_y), \tag{20}$$

$$\frac{\partial M_y}{\partial t}=g\,H\frac{\partial \zeta}{\partial y}-f\,M_x+\tau_y - \frac{g}{4\alpha A}\left(-C\frac{\partial \zeta}{\partial x}+B\frac{\partial \zeta}{\partial y}\right)-\frac{1}{A}(-E\tau_x+D\tau_y). \tag{21}$$

This system together with the equation of continuity (6) can be applied to describe the storm surge phenomenon.

In case one would like to use (20) and (21) for tidal motion, the stresses τ_x and τ_y should be replaced by the tide generating forces.

BOTTOM STRESS FORMULATION-NONSTEADY CASE

In this paragraph we shall explore a possibility to employ the Ekman time-dependent equations to describe the bottom stresses in the nonsteady equations of mass transport (7) and (8). The problem was already tackled by Jelesnianski (1970) analytically. We shall approach the problem by utilizing difference-differential equations. The set (1) and (2) will be written in differential form in space (along z), but in time it will be written in a difference form. The time axis is divided into intervals T, with $t=l$ T, $l=0, 1, 2, \ldots$

A variety of schemes to integrate equations (1) and (2) in time is available. We choose the following scheme:

$$\frac{u^{l+1}-u^l}{T}=g\frac{\partial \zeta^{l+1/2}}{\partial x}+f v^{l+1/2}+\mu\frac{\partial^2 u^{l+1}}{\partial z^2}, \tag{22}$$

$$\frac{v^{l+1}-v^l}{T}=g\frac{\partial \zeta^{l+1/2}}{\partial y}-f u^{l+1/2}+\mu\frac{\partial^2 v^{l+1}}{\partial z^2}. \tag{23}$$

Approximating $u^{l+1/2}$, $v^{l+1/2}$ by the trapezoidal rule

$$u^{l+1/2}=\frac{1}{2}(u^{l+1}+u^l), \tag{24}$$

$$v^{l+1/2}=\frac{1}{2}(v^{l+1}+v^l), \tag{25}$$

allows an explicit formulation of the semi-implicit scheme (22) and (23) which has also been used by Simons (1973).

After some algebra we arrive at the differential equation

$$\frac{\partial^2 s^{l+1}}{\partial z^2}-\alpha_T^2 s^{l+1}=Q_T^l, \tag{26}$$

where

$$s^{l+1}=u^{l+1}+iv^{l+1}, \tag{26 a}$$

$$\alpha_T^2=\frac{1+i\varepsilon}{\mu T}, \qquad \varepsilon=\frac{fT}{2}, \tag{26 b}$$

$$Q_T^l=-\frac{g}{\mu}\left(\frac{\partial \zeta^{l+1/2}}{\partial x}+i\frac{\partial \zeta^{l+1/2}}{\partial y}\right)-\alpha_T^{*2} s^l. \tag{26 c}$$

Formally (26) is the same equation as in the steady Ekman problem (10). However, α is replaced by the quantity α_T and Q is replaced by Q_T^l, thus incorporating the time step T.

Together with the boundary conditions (4) and (5) which are reformulated in complex form as

$$s^{l+1}=0 \quad \text{at} \quad z=H(x, y), \tag{27 a}$$

$$\frac{\partial s^{l+1}}{\partial z}=-\frac{\tau}{\mu} \quad \text{at} \quad z=0, \tag{27 b}$$

we are able to solve (26) analytically. Formally, the solution is the same as in the steady Ekman case (11). In this context we prefer a complex notation of the solution which, after some algebra, leads to

$$s^{l+1}=\frac{\tau^{l+1}}{\mu}\frac{\operatorname{sh}\alpha_T(H-z)}{\operatorname{ch}\alpha_T H} +\frac{g}{\mu\alpha_T^2}\left(1-\frac{\operatorname{ch}\alpha_T z}{\operatorname{ch}\alpha_T H}\right)\left(\frac{\partial \zeta}{\partial x}+i\frac{\partial \zeta^{l+1/2}}{\partial y}\right) +\frac{\alpha_T^{*2}}{\alpha_T^2}\left\{\frac{\operatorname{sh}\alpha_T(H-z)}{\operatorname{ch}\alpha_T H}\int_0^z \alpha_T s^l(\xi)\operatorname{ch}\alpha_T\xi\,d\xi +\frac{\operatorname{ch}\alpha_T z}{\operatorname{ch}\alpha_T H}\int_z^H s^l(\xi)\operatorname{sh}\alpha_T(H-\xi)\,d\xi\right\}, \tag{28}$$

with α_T^* the complex conjugate of α_T. As in the steady Ekman approach (11), the above equation (28) shows the influence of the wind stress and sealevel slope, but here a time level is involved additionally. Furthermore, the time history of the current is represented by the integral terms.

With the definition of complex bottom stress

$$\tau_b=\tau_{b,x}+i\,\tau_{b,y}, \tag{29}$$

we compute from (28):

$$\tau_b^{l+1}=-\mu\left.\frac{\partial s^{l+1}}{\partial z}\right|_{z=H} =\frac{\tau^{l+1}}{\operatorname{ch}\alpha_T H}+\frac{g}{\alpha_T}\operatorname{th}(\alpha_T H)\left(\frac{\partial \zeta}{\partial x}+i\frac{\partial \zeta}{\partial y}\right)^{l+1/2} +\frac{\mu\alpha_T^{*2}}{\operatorname{ch}\alpha_T H}\int_0^H s^l\operatorname{ch}(\alpha_T z)\,dz, \tag{30}$$

which shows that the bottom stress at time $l+1$ depends on the wind stress (at time $l+1$) and the sealevel (at time $l+(1/2)$) as is expected from our knowledge of the steady Ekman solution (16). However (30) reveals additionally that the bottom stress is dependent on the time history of the current [third term in (30)]. Different from Jelesnianski (1970) the bottom stress formulation here needs only the knowledge of the current one time step backward and of the sealevel slope half a step backward.

As in (20), (21) we can insert the bottom stress formulation (30) into the mass transport equations and on the other hand the mass transport equations can be used to predict the next sealevel slope in (22) and (23).

Since the integrals in (28) and (30) have to be evaluated numerically at each time step l and possibly at each horizontal point x, y in the region of interest, a considerable amount of computational work has to be performed. The problem of sealevel description is dependent only on the mean current, therefore, it may be worthwhile to approximate s^l in (30) by the averaged current $\bar{s}^l$ which we can get from the mass transport M_x^l, M_y^l:

$$\bar{s}^l = \frac{1}{H}(M_x + i M_y). \tag{31}$$

Replacing s^l in (30) by $\bar{s}^l$ allows us to forget the computation of (28) and to compute the integral in (30) analytically. This leads to

$$\tau_b^{l+1} = \frac{\tau^{l+1}}{\operatorname{ch} \alpha_T H} + \frac{\operatorname{th} \alpha_T H}{\alpha_T H} \times \left\{ g H \left(\frac{\partial \zeta}{\partial x} + i \frac{\partial \zeta}{\partial y} \right)^{l+1/2} + \mu \alpha_T^{*2} (M_x + i M_y)^l \right\}, \tag{32}$$

which is a simple expression compared to (28), (30). It still involves the time history of mass transport and sealevel slope for one level backward.

For comparison we rewrite the steady state bottom stress (16) in a slightly different form

$$\tau_b^{l+1} = \frac{\tau^{l+1}}{\operatorname{ch} \alpha H} + \frac{\operatorname{th} \alpha H}{\alpha H} \times \left\{ g H \left(\frac{\partial \zeta}{\partial x} + i \frac{\partial \zeta}{\partial y} \right)^{l+1/2} \right\}, \tag{33}$$

where the indexing is taken to correspond to (32).

COMPARISON OF DIFFERENT FORMULATIONS OF BOTTOM STRESS

We shall take the system (6), (7) and (8) in the numerical form used by Simons (1973). Essentially, the sealevel slope from this system will be introduced into (22) and (23). On the other hand, the stress derived from (22) and (23) is inserted into the mass transport equations. Aside of this method we will apply conventional formulations of the bottom stress in the model basin. Together we have 6 cases of bottom stress formulation to look at:

(A) Linear friction

$$\tau_b^{l+1} = \frac{r_1}{H}(M_x + i M_y). \tag{34}$$

(B) Quasi-linear friction (Platzman, 1963):

$$\tau_b^{l+1} = \frac{2.5\mu}{H^2}(M_x + i M_y). \tag{35}$$

(C) Quadratic friction (Hansen, 1956):

$$\tau_b^{l+1} = \frac{r_3}{H^2}|M_x + i M_y|(M_x + i M_y). \tag{36}$$

(A)-(C) have been discussed by Simons (1973).

(D) Friction from steady Ekman theory (Felzenbaum, 1960) as described in equation (33).

(E) Nonsteady Ekman formulation with approximation of mean current as described by equation (32).

(F) Full nonsteady Ekman formulation as described by equations (28) and (30).

Our test area will be a rectangular basin with the x-dependent topography shown in Figure 1. The subdivision in x-direction is 9×10 km, in y-direction 19×10 km. We study the sealevel and mass transport response to a sudden imposed constant wind and to a sudden periodical wind, starting from rest. The eddy viscosity value is chosen according to a modified shallow water formula (Felzenbaum, 1960) given below in the section on eddy viscosity. The test basin is located at 60° North.

The coefficients chosen for (34) and (36) respectively are $r_1 = 5.0 \times 10^{-2}$ and $r_3 = 2.5 \times 10^{-3}$.

The following wind cases are studied

$$\left.\begin{aligned} &\tau_x = 0 \quad \text{for all } t, \\ &\tau_y = 2 \quad \text{for } t > 0, \end{aligned}\right\} \tag{37}$$

which is a sudden wind to the North and

$$\left.\begin{aligned} &\tau_x = 0 \quad \text{for all } t, \\ &\tau_y = 2 \sin 2\pi/P, \\ &t \quad \text{for } t > 0 \text{ and } P = 24 \text{ hours}, \end{aligned}\right\} \tag{38}$$

which is a periodic wind in North/South direction.

Figure 2 shows the sealevel due to the wind in (37) close to the South-West corner of the test basin for all friction formulations except case D.

We have found that the constant bottom stress value taken from the steady Ekman solution (case D) and introduced into the nonsteady system of equations becomes unstable after a few time steps. This is probably due to the fact that the bottom stresses and the flow are uncoupled i. e. between them exists a time shift which may cause an amplification of motion instead of a damping. On the other hand, as shown by Kazakov (1976) and as argued by Platzman (1963), the hypothesis of proportionality of the bottom stress to the wind stress is quite appropriate for a closed channel with steady wind set up.

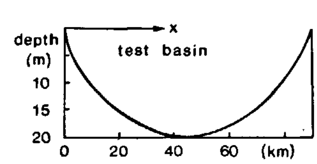

Figure 1
x−z section of the model basin.

As expected we see in Figure 2 that the sudden wind generates seiches oscillations of the sealevel. After some periods, the oscillations die off due to friction, and a steady state is reached. All friction formulations except case (F) reach the same steady state set up (13 cm). Case (F) plays a special role with respect to the steady state set up (20 cm) and to the maximum set up (29 cm). Also, the rate of damping is smaller for case (F) which in turn produces a phase shift of some hours against cases (A) to (C). It is known from comparison between model and observations that a good phase prediction is very dependent on the frictional formulations in the model. Especially, current reversals due to stress reversals are difficult to predict with respect to the phase (Kielmann, 1976). The phase differences are partly due to the change of seiches periods with friction. A linear friction increases the eigenperiods of the basin (Krauss, 1973). The friction formulation (F) tends to lengthen the eigenperiod compared to (A)-(C) although the damping is less. The unsteady Ekman friction based on a constant current profile, case (E), is the worst in terms of damping, it does not even allow a full seiches oscillation.

Figure 3 shows the sealevel response to a sudden periodic wind stress in North/South direction (period 1 day). After 3-4 periods, the response becomes sinusoidal. The amplitude of this response is highest for case (F), the lowest damping with respect to the eigenmodes of the basin is found for the quadratic friction (C), but this, as with case (A), is a matter of tuning the frictional coefficients r_1 and r_3 in (34) and (36). For all other cases the tuning is restricted to the choice of eddy viscosity which cannot be chosen arbitrarily as will be seen below. Again, different friction produces different phase behaviour of the sealevel oscillations, especially in the initial phase.

It remains to test the new formulation of bottom stress in a shallow sea like the North Sea or the Baltic under real conditions and to compare it with observations. It should be noted that a current profile is predicted in parallel with the mass transport at each point of the basin in case (F). The vertical resolution is dependent on the accuracy needed for the computation of the integrals of equation (28). The accuracy again is dependent on $\alpha_T = \alpha_T(f, \mu, T)$. In the foregoing computations we used 10 points in the vertical.

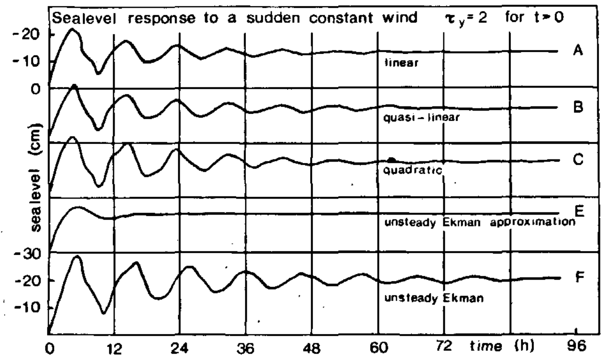

Figure 2
Sealevel response for different bottom stress formulations under a sudden imposed constant wind of about 10 m/s to the direction of positive y (North). The sealevel is taken from the South-West corner of the basin, negative sealevel means low water.

If one applies the presented method for the prediction of current profiles it may be compared to the linear part of the spectral model of Heaps (1976), with the difference, that the Heaps model uses the *a priori* bottom friction formulations as boundary conditions for the current whereas we use a non-slip condition to derive the friction from the induced current profile.

THE EDDY VISCOSITY COEFFICIENT

We shall derive the eddy viscosity coefficient only for steady state flows, but we shall use the values obtained in storm surge computations.

Let us assume that the current velocity (U_0) at the free surface is known from an empirical expression

$$\frac{U_0}{W} = k, \tag{39}$$

where W is the wind speed and $k = 0.015 - 0.020$ (Tomczak, 1964). Comparing the velocity of (11 *a*) at $z = 0$ with the velocity of (39), an unknown value of μ is obtained from the implicit equation

$$(ss^*)^{1/2}\,|_{z=0} = U_0 = W\,k. \tag{40}$$

Here s^* is the complex conjugate to s.

The implicit equation was first derived by Felzenbaum (1960) and afterwards solved numerically. We approach the problem by separating the cases of deep and shallow sea since these cases can be treated analytically (Kowalik, 1969).

Shallow sea problem ($H \to 0$, or equivalent $f \to 0$)

(11 *a*) is simplified then to

$$u = \frac{\tau_x}{\mu}(H - z) + \frac{g}{\mu}\frac{(H^2 - z^2)}{2}\frac{\partial\zeta}{\partial x}, \tag{41}$$

$$v = \frac{\tau_y}{\mu}(H - z) + \frac{g}{\mu}\frac{(H^2 - z^2)}{2}\frac{\partial\zeta}{\partial y}. \tag{42}$$

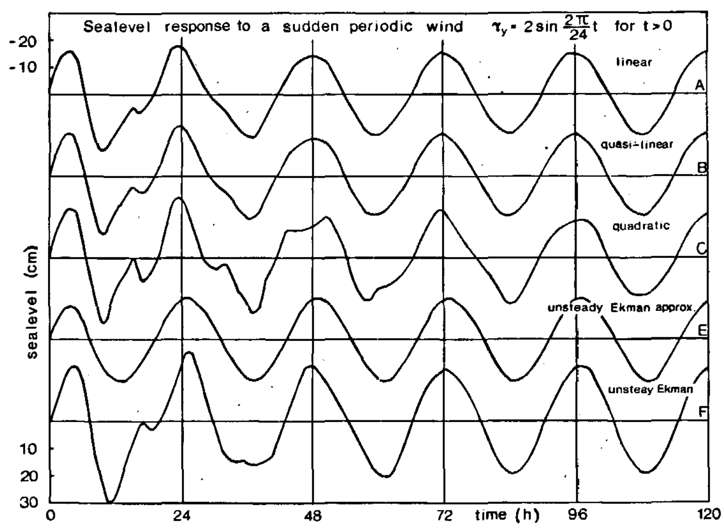

Figure 3
As figure 2, but for a periodic wind changing North-South with a period of a day and an amplitude of about 10 m/s.

Upon vertical integration of (41) and (42) and afterwards introducing a stream function through the definition (15), the sealevel slope components are obtained:

$$\frac{\partial\zeta}{\partial x}=-\frac{3}{2}\frac{\tau_x}{gH}+\frac{3\mu}{gH^3}\frac{\partial\psi}{\partial y}, \qquad (43\,a)$$

$$\frac{\partial\zeta}{\partial y}=-\frac{3}{2}\frac{\tau_y}{gH}-\frac{3\mu}{gH^3}\frac{\partial\psi}{\partial x}. \qquad (43\,b)$$

The aim of the above procedure is to introduce the components of sealevel slope into (41) and (42) so that eventually the current at the free surface will be a function of stress and some parameters but not a function of the sealevel slope.

An equation for an unknown function ψ in (43 *a*) and (43 *b*) is obtained by cross-differentiation and subsequent subtraction

$$\frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{\tau_x}{H}\right)-\frac{\partial}{\partial x}\left(\frac{\tau_y}{H}\right)\right] =\mu\left[\frac{\partial}{\partial y}\left(\frac{1}{H^3}\frac{\partial\psi}{\partial y}\right)+\frac{\partial}{\partial x}\left(\frac{1}{H^3}\frac{\partial\psi}{\partial x}\right)\right]. \qquad (44)$$

In case of τ_x, τ_y=Const., and H=Const., (44) reduces to

$$\Delta\psi=0. \qquad (45)$$

In a closed sea the boundary conditions for this problem follow from the impermeability of the coast.

Since the volume transport normal to the coast has to vanish at the coastline, the gradient of the stream-function along the coast has to vanish also, which means that ψ has to be a constant along the coastline:

$$\psi=\text{Const.}=0, \qquad (46)$$

along the coast.

An analytical function (ψ in this case) which is constant along a closed curve surrounding a simple-connected region must be constant also at all inner points (Cauchy's Integral Theorem).

With this result equations (43 *a*) and (43 *b*) have the form

$$\frac{\partial\zeta}{\partial x}=-\frac{3}{2}\frac{\tau_x}{gH};\qquad \frac{\partial\zeta}{\partial y}=-\frac{3}{2}\frac{\tau_y}{gH}. \qquad (47)$$

When (47) is introduced into (41) and (42), the components of velocity at the free surface result in

$$u_0=\frac{\tau_x H}{\mu}-\frac{3}{4}\frac{\tau_x H}{\mu}=\frac{\tau_x H}{4\mu}, \qquad (48)$$

$$v_0=\frac{\tau_y H}{4\mu}. \qquad (49)$$

Together with the empirical relation (39) we obtain

$$U_0=W k=(u_0^2+v_0^2)^{1/2} =\frac{(\tau_x^2+\tau_y^2)}{4\mu}=\frac{|\tau|H}{4\mu}. \qquad (50)$$

The connection between wind speed W and surface stress $|\tau|$ is taken as

$$|\tau|=\gamma W^2 \quad \text{with} \quad \gamma=c_D\,\rho_{air}, \qquad (51)$$

where c_D may be taken according to Garrat (1977) as

$$c_D=(0.75+0.067\,W)\times 10^{-3}, \qquad W\ (m/s). \qquad (52)$$

Solving equation (50) for μ and expressing $|\tau|$ according to (51) leads finally to

$$\mu=\frac{\gamma WH}{4k} \quad \text{for shallow water.} \qquad (53)$$

Deep sea (H → ∞)

In this case (11 *a*) can be approximated by

$$u=\frac{e^{-\alpha_1 z}}{\alpha_1\mu\sqrt{2}}\left[\tau_x\sin\left(\frac{\pi}{4}-\alpha_1 z\right)+\tau_y\cos\left(\frac{\pi}{4}-\alpha_1 z\right)\right]+\frac{g}{f}\frac{\partial\zeta}{\partial y}, \qquad (54)$$

$$v=\frac{e^{-\alpha_1 z}}{\alpha_1\mu\sqrt{2}}\left[-\tau_x\cos\left(\frac{\pi}{4}-\alpha_1 z\right)+\tau_y\sin\left(\frac{\pi}{4}-\alpha_1 z\right)\right]-\frac{g}{f}\frac{\partial\zeta}{\partial x}. \qquad (55)$$

Upon vertical integration of (54), (55) and introduction of the stream function, the sealevel slope is derived

$$\frac{\partial\zeta}{\partial x}=-\frac{\tau_x}{gH}-\frac{f}{gH}\frac{\partial\psi}{\partial x}, \qquad (56)$$

$$\frac{\partial\zeta}{\partial y}=-\frac{\tau_y}{gH}-\frac{f}{gH}\frac{\partial\psi}{\partial}. \qquad (57)$$

Applying the cross-differentiation technique and solving under the same conditions as for the shallow sea we derive the same result, i. e. $\partial\psi/\partial x=0$, $\partial\psi/\partial y=0$. Therefore, the above set simplifies to

$$\frac{\partial\zeta}{\partial x}=-\frac{\tau_x}{gH};\qquad \frac{\partial\zeta}{\partial y}=-\frac{\tau_y}{gH}. \qquad (58)$$

For H → ∞ the sealevel slopes vanish and, therefore, from (54) and (55) the current components at the sea surface ($z=0$) are

$$u_0=\frac{1}{2\alpha_1\mu}(\tau_x+\tau_y), \qquad (59)$$

$$v_0=\frac{1}{2\alpha_1\mu}(\tau_y-\tau_x). \qquad (60)$$

Using again (39) and (51) we arrive at

$$\mu=\left(\frac{\gamma}{k}\right)^2\frac{W^2}{f} \quad \text{for deep water.} \qquad (61)$$

Since (53) holds in the shallow sea and (61) in the deep one, it is possible to equalize both expressions and to derive the critical water depth

$$H_{cr}=4\frac{\gamma}{k}\frac{W}{f}, \qquad (62)$$

which divides the regions where the formulae (53) and (61) can be applied. If $H < H_{cr}$, (53) should be used. The expressions for the eddy viscosity μ, (53) for shallow water and (61) for deep water, where shallow and deep are separated by the depth H_{cr} in (62), can now be used for the general integration procedure outlined above (α_T (26 *b*) depends on μ). Thus, after having accepted that surface current and windspeed follow the empirical expression in (39), there is no longer the possibility of tuning μ to the observations.

CONCLUSION

The foregoing study shows, for a simple test case, that many parameters such as eigenperiod, water set up, phase behaviour, damping or energy dissipation, etc. are strongly influenced by the choice of bottom friction. Especially, the presented formulation which is consistent with the unsteady Ekman theory reveals different behaviour if compared e. g. with the *a priori* quadratic friction law. It remains to test the new formulation in a more realistic environment such as the Baltic or North Sea. The proposed method could also be generalized for a multilayer stratified model, where the wind stress is replaced by the stress acting on the top of the lowest layer. Since bottom stress formulations try to parameterize the processes in the bottom boundary layer, they have a strong impact on any marine ecosystem modelling in connection with physical modelling.

A deficiency of the presented bottom stress formulation is the fact that the vertical eddy viscosity μ is constant with respect to z together with a no-slip condition. Both facts do not provide for the transition of the Ekman boundary layer into a Prandtl logarithmic layer at the bottom. The limitation to a constant μ could be overcome by choosing a somehow realistic μ-profile for which equation (26) can still be solved analytically, although, due to our sparse knowledge of $\mu(z)$, there is a lot of arbitraryness involved. One could of course try to compute $\mu(z)$ independently from the energy equation as Kowalik (1972) did. A slip condition, on the other hand, would add an additional constant to tune the model, a fact, that we tried to avoid here. A major deficiency is the neglection of stratification which seriously affects the vertical current profile. Since we are dealing with a bottom stress formulation for storm surge problems we are rather interested in the vertically integrated flow than in the vertical variation of the flow.

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Glossary of symbols

f, Coriolis parameter;
g, earth acceleration;
H, water depth;
M_x, M_y, volume transport components;
$M = M_x + i M_y$, complex volume transport;
$Q = -\frac{g}{\mu}\left(\frac{\partial\zeta}{\partial x} + i\frac{\partial\zeta}{\partial y}\right)$;
r_1, r_3 = frictional constants;
$s = u + iv$, complex horizontal current;
$k = U_0/W = 0.015 - 0.020$ (empirical constant);
$\bar{s}$, $\bar{\bar{s}} = M/H$ = depth averaged velocity;
t, time;
T = time step;
u, v, w, current components in x, y, z direction;
$U_0 = (u_0^2 + v_0^2)^{1/2}$ = current speed at the surface;
W = wind speed;
x, y, z, left handed coordinate system, z downward;
$\alpha = \sqrt{i\frac{f}{\mu}}$;
$\alpha_1 = \sqrt{\frac{f}{2\mu}}$;
$\alpha_T = \sqrt{\frac{1+i\varepsilon}{\mu T}}$;
$\varepsilon = f\,T/2$;
$\gamma = c_D\,\rho_{air}$;
ψ = stream function, $M_x = \frac{\partial\psi}{\partial y}$, $M_y = -\frac{\partial\psi}{\partial x}$;
ρ_{air} = density of air;
τ_x, τ_y, kinematic wind stress components;
τ_{bx}, τ_{by}, kinematic bottom stress components;
$\tau = \tau_x + i\,\tau_y$ = complex wind stress;
$\tau_b = \tau_{bx} + i\,\tau_{by}$ = complex bottom stress;
ζ, sealevel, positive in direction of z;
c_D, drag coefficient;
μ, kinematic eddy viscosity, vertical diffusion of momentum.