Skewness of the generalized centrifugal force divergence for a joint normal distribution of strain and vorticity components

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This note attempts to connect the skewness of the probability distribution function (PDF) of pressure, which is commonly observed in two-dimensional turbulence, to differences in the geometry of the strain and vorticity fields. This paper illustrates analytically the respective roles of strain and vorticity in shaping the PDF of pressure, in the particular case of a joint normal distribution of velocity gradients. The latter assumption is not valid in general in direct numerical simulations (DNS) of two-dimensional turbulence but may apply to geostrophic turbulence in presence of a differential rotation (β effect). In essence, minus the Laplacian of pressure is the difference of squared strain and vorticity, a quantity which is named the generalized centrifugal force divergence (GCFD). Square strain and vorticity distributions follow chi-square statistics with unequal numbers of degrees of freedom, when one assumes a joint normal distribution of their components. Squared strain has two degrees of freedom and squared vorticity only one, thereby causing a skewness of the PDF of GCFD and hence of pressure.

Direct measurements of the pressure field in three-dimensional turbulence have confirmed the skewness of the distribution between regions of low and high pressure.[1] This skewness of pressure of the probability distribution function (PDF) is generally larger in the case of two-dimensional turbulence.[2] The last reference has analytically established for both two and three dimensions, that the pressure PDF has exponential tails and is skewed even for a Gaussian velocity field. In the present note, a very simple attempt is made at connecting the origin of this skewness to intrinsic differences of geometry of the strain and vorticity fields in two-dimensional turbulence. For both two and three dimensions, the pressure field is linked to the competition between strain and vorticity through the diagnostic relation

$$2\nabla^2 p = \omega^2 - s^2, \tag{1}$$

where the gradient operator ∇, vorticity ω, and strain s are appropriately defined for the dimension under consideration.

We shall only consider here the case of two-dimensional turbulence and concentrate on the PDF of the variable W

$$W = s^2 - \omega^2, \tag{2}$$

instead of the PDF of pressure.

The quantity W has received considerable attention in recent studies of turbulent transport of two-dimensional turbulence.[3–5] In the context of geostrophic turbulence, it has been proposed[6] to name W the centrifugal force divergence (since it is the divergence of a force potential and it also measures the competition between the centrifugal tendencies of strain deformation and the trapping tendencies of spin within the vortex cores), and some of its spectral characteristics are given in Ref. 6.

By definition, the strain variance corresponds to the sum of the variances of two components, shear and normal strain ($s^2 = s_1^2 + s_2^2$), while vorticity corresponds to a single degree of freedom. We have

$$s_1 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -2\,\frac{\partial^2 \psi}{\partial x\, \partial y},$$

$$s_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\psi,$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi,$$

where ψ is the streamfunction of the flow. One can also account for unequal numbers of degrees of freedom in strain and vorticity by noting that the symmetric rate-of-strain tensor contains two independent terms (when taking into account incompressibility), while the antisymmetric vorticity tensor has only one nonzero term.

For a homogeneous turbulence, the three variables s_1, s_2, and $(1/\sqrt{2})\omega$ are uncorrelated and have the same variance for a spatial ensemble average:

$$\langle s_1 s_2\rangle = \langle s_1 \omega\rangle = \langle s_2 \omega\rangle = 0,$$
$$\langle s_1^2\rangle = \langle s_2^2\rangle = \tfrac{1}{2}\langle \omega^2\rangle = \sigma^2. \tag{3}$$

This suggests to examine the implications of the simplest assumption about their statistics which is a joint normal distribution of the three variables. We emphasize that direct numerical simulations (DNS) of two-dimensional turbulence show that a joint normal assumption is not valid in general, but the main purpose of the following derivation is to examine the implications of unequal degrees of freedom of strain and vorticity on the the PDF of W, using an analytically straightforward framework.

The chi-square statistics of the quantity $y = \chi^2$ defined as

$$\chi^2 = x_1^2 + x_2^2 + \cdots x_n^2, \tag{4}$$

where the random variables $x_1, x_2, \ldots$ are normal and independent with the same variance σ^2, is given by the density function[7]

$$f_y(y) = \frac{1}{2^{n/2}\sigma^n \Gamma(n/2)}\, y^{(n-2)/2} e^{-y/2\sigma^2} H(y), \tag{5}$$

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where $H(y)$ is the Heaviside function. The distributions of squared strain and vorticity correspond respectively to $n=2$ and $n=1$ in (5) and are thus

$$f_x(x=s^2)=\frac{1}{2\sigma^2}\,e^{-x/2\sigma^2}H(x),$$
$$f_y(y=\omega^2)=\frac{1}{\sigma}\,\frac{1}{2\sqrt{\pi y}}\,e^{-y/4\sigma^2}H(y). \tag{6}$$

Thus even in the case of joint normal distribution of velocity gradients, the PDFs of squared vorticity and squared strain are quite distinct because of their unequal number of degrees of freedom. Positive values of W are dominated by strain statistics while negative values correspond to large vorticity, thereby leading to a skewed distribution.

We note the PDFs that appear in (6) are identical to the generic forms of PDF tails found by Ref. 2, when the leading singularity of the pressure generating function is either a square root or a simple pole. The PDF of $W=x-y$ is readily obtained from the PDF of x and y:

$$f_W(W)=\int_{-\infty}^{+\infty} f_x(W-y)f_y(-y)dy.$$

We have

$$f_W(W)=\frac{1}{2\sqrt{3}\sigma^2}\,e^{-W/2\sigma^2},\quad W>0,$$
$$f_W(W)=\frac{1}{2\sqrt{3}\sigma^2}\,e^{-W/4\sigma^2}\left[1-\mathrm{erf}\left(\sqrt{\frac{3|W|}{2\sigma^2}}\right)\right],\quad W<0, \tag{7}$$

and the moments of W are

$$\langle W\rangle=0,$$
$$\langle W^2\rangle=12\sigma^4,$$
$$\langle W^3\rangle/\langle W^2\rangle^{3/2}=-1.15, \tag{8}$$
$$\langle W^4\rangle/\langle W^2\rangle^2=9.$$

Figure 1 illustrates the PDF of W found in a direct numerical simulation of geostrophic turbulence (continuous line)[6] for a resolution of $(256)^2\times 4$ and for a case of moderate intermittency, as measured by the kurtosis of vorticity which is of 3.4. On the same figure are displayed the analytical expression (7) of the PDF (dashed line) and a Gaussian PDF (dotted line) which variance is equal to (8b). For the numerical simulation of Fig. 1, we find a skewness of -1.33 and a kurtosis of 13, which are reasonably close to the values given in (8). The example given in Fig. 1 stems form a numerical realization of geostrophic turbulence,[6] in presence of a differential rotation (β effect) which significantly curbs the level of intermittency of this geophysical example of nearly two-dimensional turbulence. The numerically observed low level of intermittency actually motivated the present study of the PDF of W for a joint normal distribution of strain and vorticity components. Numerical simulations with increasing levels of intermittency (corresponding to weaker β effects) reveal that the amplitudes of both the negative skewness and

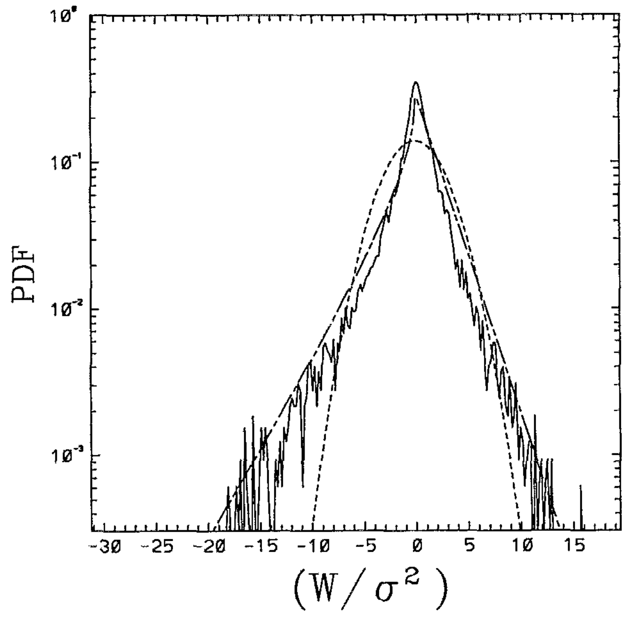

FIG. 1. PDFs of the generalized centrifugal force divergence $W=-\nabla^2 p$: (a) as observed in a numerical simulation with small intermittency (continuous line); (b) as given by analytical expression (7) (dashed line); (c) for a Gaussian distribution with equal variance (dotted line). The abscissa is normalized by σ^2 which is defined by (3).

kurtosis continuously grow, while tails of the PDF progressively become longer and extend beyond the analytical PDF given in (7). Intermittency appears to enhance the statistical trends already present for a joint normal distribution of velocity gradients. Furthermore, strong intermittency can dramatically amplify these tendencies, e.g., we found for a simulation with a kurtosis of 50 of vorticity that the skewness and kurtosis of W were, respectively, -23 and 1100.

In summary, we have shown that even for the case of a joint normal distribution of velocity gradients, the distribution of the generalized centrifugal force divergence (GCFD) will be skewed because of differences in dimensions of the strain and vorticity fields in two-dimensional turbulence. The GCFD is a reduced quadratic form ($W=s_1^2+s_2^2-\omega^2$) which has a signature of (2,1), i.e., it has two positive squared terms and one negative square term. This property ensures that W will have exponential tails and a skewed distribution. The simple arguments given above do not generalize straightforwardly to the case of three dimensions, since squared strain no longer follows a chi-statistic of independent variables, even when one assumes a joint normal distribution of velocity gradients. This results from the incompressibility constraint ($\Sigma_{n=1}^3 s_{ii}=0$, s_{ij} being the rate-of-strain tensor) and the GCFD is thus no longer a reduced quadratic form of independent variables for a three-dimensional flow.

Finally, the skewness of the pressure field p is directly connected to the skewness of W in two dimensions. This is readily shown by using the analytical results of Ref. 2 for Gaussian velocity fields. Their relation (18) expresses the third moment of p as an integral in Fourier wave-number space of a negative definite term:

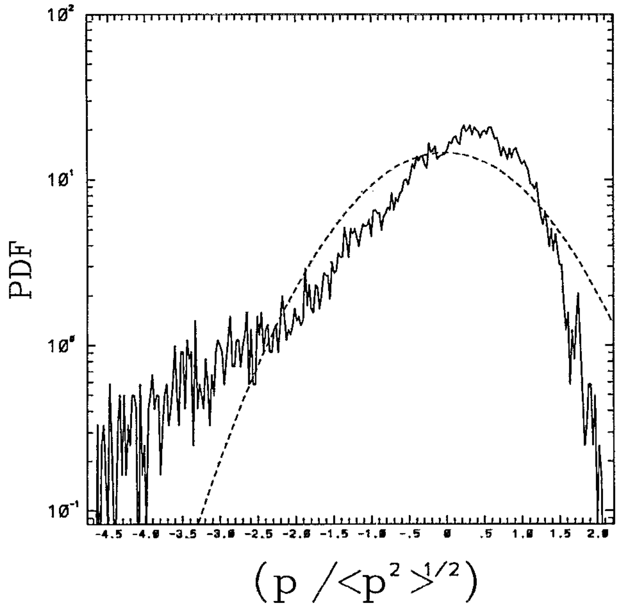

FIG. 2. Pdfs of the pressure field p: (a) as observed in the same numerical simulation than Fig. 1(a) (continuous line); (b) for a Gaussian distribution with equal variance (dotted line).

$$\langle p^3\rangle = -8\int (d\mathbf{k}_1)(d\mathbf{k}_2)(d\mathbf{k}_3) f(k_1)f(k_2)f(k_3) \times k_1^2k_2^2k_3^2 \frac{\sin^2(\theta_{12})}{(\mathbf{k}_1-\mathbf{k}_2)^2}\frac{\sin^2(\theta_{23})}{(\mathbf{k}_2-\mathbf{k}_3)^2}\frac{\sin^2(\theta_{31})}{(\mathbf{k}_3-\mathbf{k}_1)^2}, \quad (9)$$

where $f(k)\propto E(k)/k$, $E(k)$ being the spectrum of the turbulence. Since relations (1) and (2) imply in Fourier space that $W(\mathbf{k})=+\mathbf{k}^2p(\mathbf{k})$, an analogous computation for $\langle W^3\rangle$ would produce as integrand the numerator of the integrand of (9), which would therefore also be negative definite. Thus one can state that the skewness of pressure distribution is essentially a consequence of the skewness of W. Figure 2 illustrates the pressure PDF found in the same numerical simulation than that of Fig. 1(a), and for which the skewness of p is -1.4.

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