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2D vortex interaction in a non-uniform flow

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Abstract:

In a two-dimensional incompressible fluid, we study the interaction of two like-signed Rankine vortices embedded in a steady shear/strain flow. The numerical results of vortex evolutions are compared with the analytical results for point vortices. We show the existence of vortex equilibria, and of merger for initial distances larger than those without external flow. The evolutions depend on the initial orientation of the vortices in the external flow.

Keywords: Two-dimensional incompressible flows - Vortex merger - Pseudo-spectral model

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1 Introduction

Vortices play an essential role in 2D turbulent fluids where they achieve the energy cascade towards larger scales, the enstrophy cascade towards small scales being carried out via filaments. These vortices and filaments are often the end-product of the merger of smaller vortices. The merging process has often been studied, in particular with piecewise-constant (Rankine) vortices [6,9,13,17]. In 2D incompressible and inviscid flows, the merger of two equal Rankine vortices occurs if the two vortex centers are initially closer than 3.2 radii. This critical distance is modified by other factors, such as unequal size of vortices [12,18], flow divergence [1], barotropic instability of the vortices [4] or beta effect [2]. Recently, theoretical studies considered the interaction of two point vortices with steady or unsteady external flow [5,10,11,14,15]. Here, we use their results to interpret numerical simulations of the interaction of two identical Rankine vortices, in steady deformation fields. Firstly we recall the analytical results for point vortices (Sect. 2). In Sect. 3, the interaction of two Rankine vortices in a steady external flow is modeled with a spectral model. Finally conclusions are drawn.

2 The analytical results

In an inviscid, incompressible and homogeneous 2D fluid, motions are governed by the vorticity equation:

$$\partial_t \zeta + J(\psi, \zeta) = 0$$

where $\zeta = \nabla^2\psi$ is relative vorticity, ψ is streamfunction and J is the Jacobian operator.

For the analytical study, we consider two point-vortices with circulation Γ, and an external flow composed of global rotation (with rate $\Omega(t)$) and of strain (with rate $S(t)$). The external flow and the vortex pair have central symmetry. The vortex positions with respect to the center of the plane are given by their radius ρ and by their angle θ (resp. $\theta + \pi$). The equations of motion are, for any of the two vortices:

$$v_r^* = \dot{\rho} \;=\; -\frac{1}{\rho}\frac{\partial H}{\partial \theta} \;=\; -\; s(t)\rho\sin(2\theta) \tag{1a}$$

$$v_\theta^* = \rho\dot{\theta} \;=\; \frac{\partial H}{\partial \rho} \;=\; \frac{1}{4\rho} \;+\; \rho\omega(t) \;-\; s(t)\rho\cos(2\theta) \tag{1b}$$

where we have set $v_r^* = \pi v_r/\Gamma$, $v_\theta^* = \pi v_\theta/\Gamma$,
$s(t) \;=\; \pi S(t)/\Gamma \;=\; s_0\,(1 \;+\; \varepsilon\delta\cos(\sigma t))$,
$\omega(t) \;=\; \pi\Omega(t)/\Gamma \;=\; \omega_0\,(1 \;+\; \varepsilon\delta\cos(\sigma t))$, $|\varepsilon| \ll 1$, $|\delta| \sim 1$.
where ε is a small parameter for the expansion and δ represents the relative contribution of the unsteady part of the flow.
A normalized Hamiltonian corresponds to this system $H(\rho,\theta,t) \;=\; \frac{1}{4}\,[\ln(2\rho) - 2s(t)\rho^2\cos(2\theta) + 2\omega(t)\rho^2]$
In the expansion of the equations of motion in ε, the zeroth order $\partial_t\rho_0 \;=\; \partial_t\theta_0 \;=\; 0$ provides the equilibria (at most four, related by symmetry). Their existence and position depend on the values of the steady strain and rotation rates (s_0 and ω_0):

$$\theta_0 = n\pi/2, \quad \rho_0 = \frac{1}{2\sqrt{(-1)^n s_0 - \omega_0}}$$

The stability of the equilibria is given by the first order equations :

- If $-\omega_0 > |s_0|$, the equilibria are two neutral points and two saddle-points.
- If $|s_0| > |\omega_0|$, they are two saddle-points.
- And if $\omega_0 > |s_0|$, there is no equilibrium.

Adding an unsteady component to the external flow leads to the appearance of a modulation in the oscillation around the neutral point. This modulation can be studied by a multiple time scale expansion.This addition also leads to the appearance of Hamiltonian chaos in the vicinity of the heteroclinic curves (due to their mutual crossing, as shown by Melnikov's method and via Poincaré sections). Chaos extends in the Poincaré section when δ increases ([15]).

3 Interaction of two finite-area vortices in a steady external flow

Our numerical study focuses on vortex evolutions when $-\omega_0 > |s_0|$ (case of neutral point existence). A pseudo-spectral model of 2D incompressible flows is initialized with two circular vortices with radius r, unit vorticity and intercentroid distance d. The vortices can be located either on the axis of the neutral points, or on that of the saddle points. We set $\omega_0 = 2s_0$.

3.1 Vortices initially on the axis of the neutral points

In this case, the vortices can remain near the neutral points. We explore the dynamical regimes for $d/r \in [2.3, 10]$, and $\omega_0 \in [-0.09, -0.01]$. The results are summarized in Fig:1
For weak external flow and small d/r, the regimes are the same as without external flow: merger and anticlockwise rotation. When the external flow amplitude increases, it acts against merger at small d/r. Merger is then replaced by vortex interaction with filament shedding, followed by strain-induced expulsion when the vortices have decreased in size.
For moderate external flow and larger d/r, vortices can reach stationary positions, which agree with the neutral positions of point vortices, determined analytically. In the $(d/r, \omega_0)$ plane, the vicinity of this stationary case corresponds to vortex oscillations around the neutral point position. For values of d/r smaller than those of the fixed point, these oscillations start anticlockwise because co-rotation dominates. On the contrary, for d/r larger than that of the fixed point, they start clockwise because

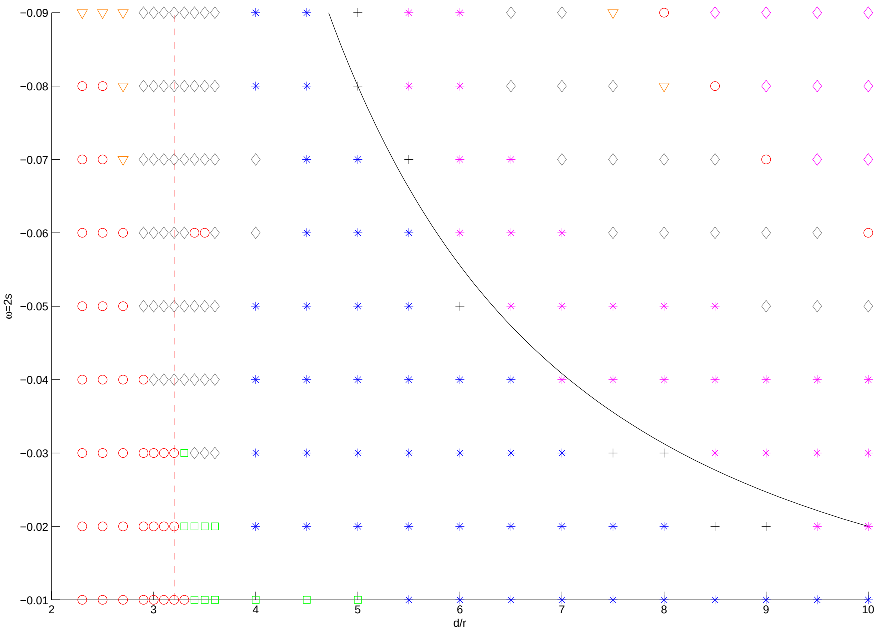

Fig. 1 The different regimes for vortices initially on the neutral point axis; red circle=merger, black cross=fixed point, green square=anticlockwise rotation, blue/magenta star=oscillation with initial anticlockwise/clockwise motion, grey diamond=expulsion after destructive interaction, orange triangle=straining out, magenta diamond=expulsion, solid black line =positions of the neutral point, as determined by the analytical study of point vortices, red dashed line =limit of merger when there is no external flow.

the external flow dominates.

For still larger external flow and d/r, vortex interaction leads to filament shedding, erosion and eventually, the vortices are expelled along the extension axes of the external strain.

Merger is again observed for $\omega \sim -0.08$ and large d/r. To the best of our knowledge, this is the first instance where such merger is reported. Fig:2 shows that, in this case, vortices move along the separatrices, thereby reducing their distance and finally, merger is allowed.

When the external strain is intense, vortices are elongated until they are destroyed, as Kida ([7]) showed, for an elliptical vortex in a strong external shear flow. As in Kida's study, we observe stationary vortex positions and vortex rotation for weak external strain. But we do not observe vortex nutation. Note that here, contrary to Kida's study, vortices are not centered in the strain, and a finite value of strain-induced velocity always exists at the vortex center. For strong external strain, the induced vortex deformation dominates their tendency to merge and leads to vortex destruction, as seen in Fig:3.

The elliptical vortex model, which assumes that vortices remain elliptical at all times ([9]), is used for comparison with the pseudo-spectral model results. In the elliptical model, the merging criterion is based on vortex contact. The elliptical model results shown in Fig:4 show satisfactory agreement with the pseudo-spectral model results for vortex merger.

3.2 Vortices initially along the axis of the saddle points

In this case, the vortices cannot remain near their initial positions because the saddle-points are unstable. The dynamical regimes are investigated for $d/r \in [3.2, 8]$ and for $\omega_0 \in [-0.09, -0.01]$. The results are summarized in Fig:5.

For weak external flow and for small d/r, the dynamical regimes are merger and co-rotation, as for vortices initially near the neutral points. Again for small d/r, an increase in external flow first favors

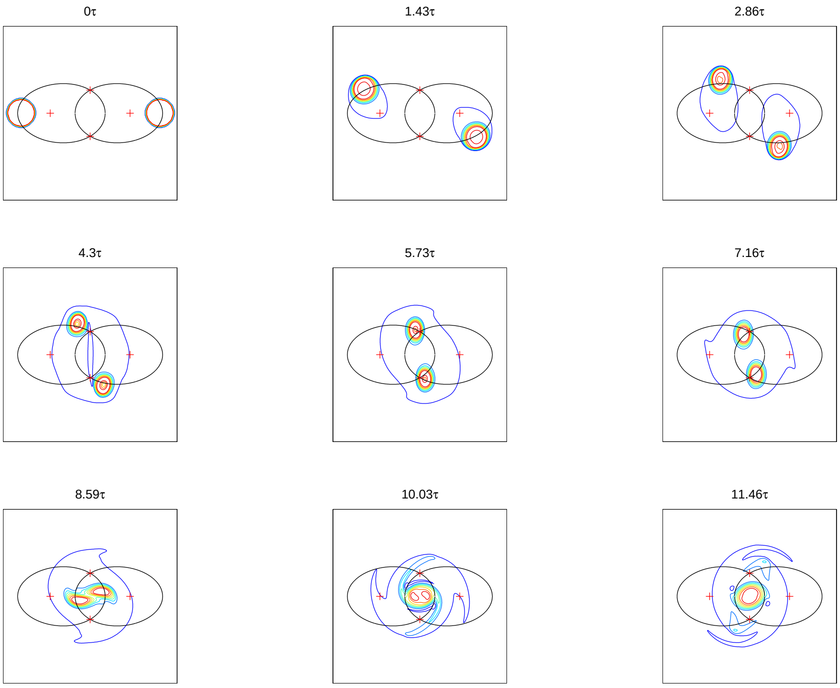

Fig. 2 The time evolution of two vortices initially located near the neutral points and near the separatrices (black solid line); the two red crosses on the separatrices are the saddle points and the two others correspond to the neutral points; the colored contours are vorticity contours.

merger and then acts against it. There is also a reversal in the initial direction of rotation of the vortices.
Indeed, for close vortices, increasing the external strain first tends to bring them closer to the center and to each other. But for large external strain, the vortices become located beyond the saddle-points, and are then expelled. The position of the saddle-points thus found numerically agrees with the analytical value (for point-vortices) when it is farther than $3.3r$ from the center. The direction of rotation is governed by vortex interaction at small distances and by external strain at large distances.
For stronger external flow, the straining out of vortices leads to their destruction.
In summary, this case is more simple than the former, due to the absence of steady vortex position.

4 Conclusion

Numerical simulations of two equal vortex interaction under external strain and rotation confirm the analytical study for point vortices : the fixed points, both stable and unstable, are influential on the interaction of finite-area vortices. If the vortices are initially located on the axis of neutral points, the dynamical regimes in the parameter plane are varied and well explained by the relative influence of their companion vortex (influence of d/r) and of the external flow (influence of s and ω). An essential result obtained here is the possibility for two initially distant vortices to merge, a process allowed by the external flow. If the vortices lie initially on the axis of the saddle points, the regime diagram is simpler, but again, evidences the competition between these two influences. An analysis of vortex deformation with the Okubo-Weiss criterion (see previous work by [8], [16], [3]), distinguishing vortex-induced strain and external strain, will be achieved to better understand the influence of vortex erosion during their interaction. A generalization of this study to stratified flows will be undertaken for oceanic applications.

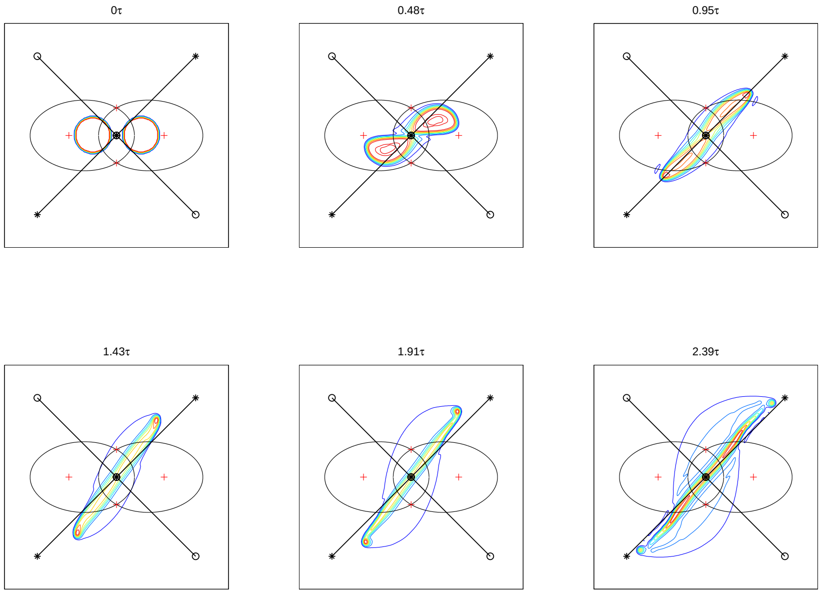

Fig. 3 The straining out of a vortex pair; the two lines correspond to the strain axes strain (extension axis with stars, compression axis with circles. The black ellipses, the red crosses and the colored contours are as in figure2.

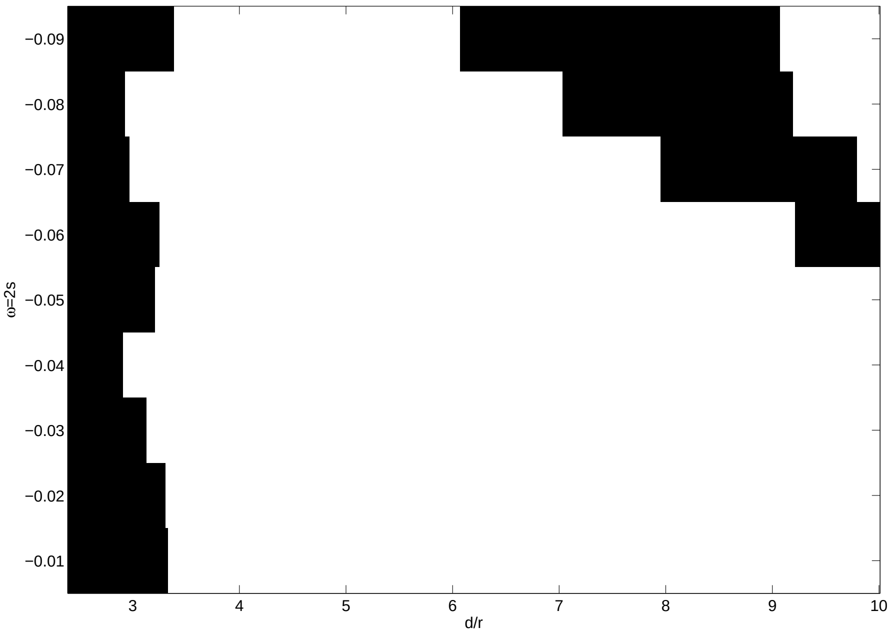

Fig. 4 Dynamical regimes for elliptical vortices: in black, regions of vortex merger (vortices get in contact); in white, distant interactions

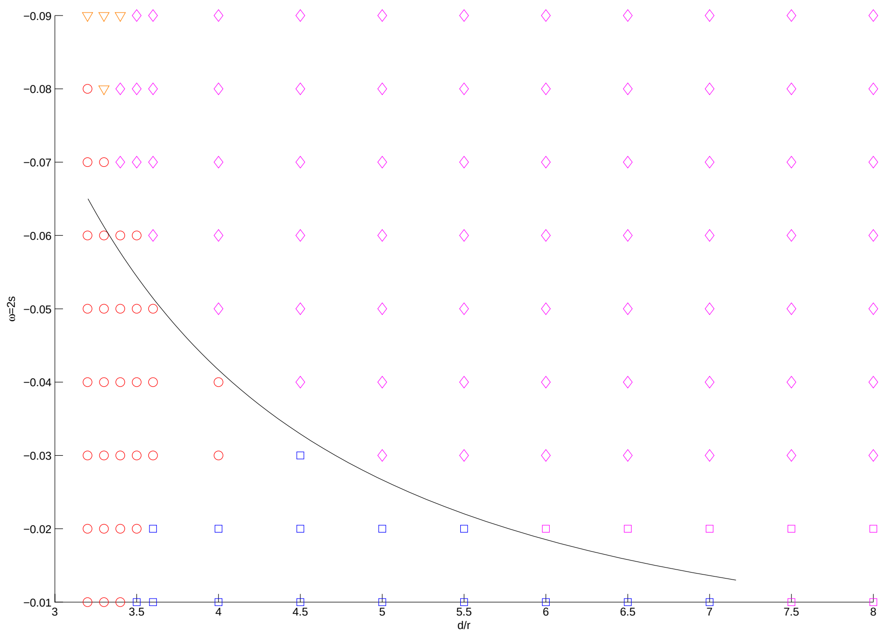

Fig. 5 The different regimes for vortices initially along the axis of the saddle points; red circle=merger, green/magenta square=anticlockwise/clockwise rotation, orange triangle=straining out, magenta diamond =expulsion, solid black line=analytical positions of the saddle point.

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