Short Communication

A continuous hockey stick stock–recruit model for estimating MSY reference points

Benoit Mesnil and Marie-Joëlle Rochet

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With political commitment to restore stocks to levels where they can produce maximum sustainable yield (MSY), fisheries managers request evaluation of management plans that include options for an F_{MSY} policy. The procedure to estimate F_{MSY} with dynamic-pool, stock assessment models is well established for common stock–recruitment relationships (S–RR), and this capacity is extended to another S–RR, a piecewise function known as the hockey stick (HS), which is frequently assumed when the data do not support more elaborate functions. However, the HS is not continuous, which makes it problematic for this application, where differentiable functions are required. The bent-hyperbola model proves to be an adequate continuous equivalent to the HS for estimating F_{MSY}.

Keywords: bent hyperbola, F_{MSY}, hockey stick, MSY, stock–recruitment.

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B. Mesnil and M-J. Rochet: Ifremer, Département EMH, BP 21105-44311, Nantes Cedex 3, France. Correspondence to B. Mesnil: tel: +33 240 37 40 09; fax: +33 240 37 40 75; e-mail: benoit.mesnil@ifremer.fr.

Introduction

In the FAO fisheries glossary (http://www.fao.org/fi/glossary/default.asp), maximum sustainable yield (MSY) is defined as “the highest theoretical equilibrium yield that can be continuously taken (on average) from a stock under existing (average) environmental conditions without affecting significantly the reproduction process”. This definition emphasizes that MSY not only addresses maximization of yield (e.g. as in yield-per-recruit analyses), but also targets preservation of a stock at a sufficiently high level to reproduce itself. With age-structured analytical (also known as dynamic-pool) models used in most ICES assessments, estimating MSY reference points implies that yield-per-recruit analyses need to be combined with stock–recruitment relationships (S–RR), using a “classical” procedure involving the concept of replacement line and its intercept with the S–RR curve (Sissenwine and Shepherd, 1987; Quinn and Deriso, 1999).

Most often, however, plots of stock and recruit estimates from fish stock assessments provide equivocal indications on the precise form of the underlying S–RR, resulting in large uncertainties on the plausible location of F_{MSY}. Yet, in many applications such as bioeconomic modelling (Clark *et al.*, 1985), evaluations of management strategies and management plans (e.g. Daan, 2007), and identification of precautionary reference points (ICES, 2003), the need arises to take account of the essential feature of the S–RR (“the sensible null hypothesis that recruitment is likely to fall at low SSB”; Shepherd, 1982) and some degree of compensation. This assumption is supported by empirical evidence, across taxonomic groups, that recruitment tends to be poorest when spawner abundance is low (Myers and Barrowman, 1996).

A simple depiction capturing these features is a piecewise relation, where recruitment is set at the average value for all spawner-stock biomasses (SSBs) above some threshold and is linearly reduced towards zero as SSB approaches zero (Clark *et al.*, 1985; Butterworth and Bergh, 1993). This is also known as a segmented regression, but a popular name is the “hockey stick (HS)” S–RR (Barrowman and Myers, 2000). A problem with the HS S–RR is its piecewise formulation, which makes estimating MSY and the associated fishing mortality F_{MSY} impossible with the “classical” procedure alluded to above. A way around the problem is to consider a continuous approximation of the HS, such as the bent hyperbola (Watts and Bacon, 1984). The properties of the bent hyperbola and the procedure to estimate F_{MSY} in combination with this S–RR, with its pros and cons, are discussed here.

Material and methods

Derivation of F_{MSY} with dynamic-pool and parametric S–RR models

A quick reminder is provided of the procedure described in the literature (Laurec and Le Guen, 1981; Shepherd, 1982; Sissenwine and Shepherd, 1987; Quinn and Deriso, 1999). S is used as the symbol for SSB or any other appropriate metric of effective fecundity. One starts with a per-recruit analysis which, for each fishing mortality (F) value, provides a single estimate of yield (Y/R), and one of spawning biomass (S/R). It is common to use a discrete yield model, with this set of equations:

$$N_i = R\exp\left\{-\sum_{j=0}^{i-1}\left(s_jF + M_j\right)\right\}; \quad R = 1,$$

$$C_i = \frac{s_i F}{s_i F + M_i}\{1 - \exp(-s_i F - M_i)\}N_i,$$

$$Y/R = \sum_i C_i W_i,$$

$$S/R = \sum_i N_i O_i W_i,$$

where i is the age, N_i the survivors at age i from a recruitment of size R (typically unity), F a nominal fishing mortality and s_i its distribution-by-age (selection pattern), M_i the natural mortality, C_i the catch-at-age i in number, W_i the weight-at-age, O_i the fraction mature at age, Y the yield in weight, and S the spawning biomass. For a given F, there is one value of S per recruit, noted λ_F hereafter.

Next, one turns to the graph of the S–RR function $R = \phi(S)$. The line through the origin of slope $1/\lambda_F$, referred to as the replacement line, cuts the S–RR curve ϕ for some S abcissa (getting smaller as F increases, because the replacement line becomes steeper). At equilibrium (the same R gives the same S, which gives the same R), the following condition applies for all S–R models:

$$S_e = \lambda_F \times \phi(S_e). \tag{1}$$

If ϕ^{-1} exists, as it does for functional forms of S–RR, this equation can be solved for S_e, and the recruitment at long-term equilibrium is obtained as $R_e = S_e/\lambda_F$. For that value of F, the equilibrium yield $Y_e = R_e \times Y/R$.

All this holds for a single value of F (or F-factor). If we repeat the calculation over an appropriate range of Fs, we obtain a curve of Y_e, which peaks for a given value of F; this F is the desired estimate of F_{MSY}, and the peak Y_e is MSY. The corresponding S_e is B_{MSY}, the spawner biomass producing MSY. F_{MSY} can also be estimated analytically as the fishing mortality where the derivative of the yield curve is zero; hence, the need for a differentiable S–RR function.

The HS and its continuous variants

The HS S–RR (Butterworth and Bergh, 1993; Barrowman and Myers, 2000) is a segmented function whose curve starts with slope $a > 0$ at the origin and then becomes horizontal beyond some level of spawning abundance, S^*:

$$R = \begin{cases} aS, & S < S^* \\ R^* = aS^*, & S \geq S^* \end{cases}. \tag{2}$$

On a plot, the curve bends sharply at the breakpoint S^*. This is known to cause mathematical difficulties for inference (likelihood surface with flat ridges). For our purpose, the main problem is that there is no inverse function allowing a solution to Equation (1), so a continuous analogue is needed.

To cope with inference problems, Barrowman and Myers (2000) proposed a smooth variant, the logistic HS (LHS), whose properties were further analysed by Cadigan and Healey (2004). However, Cadigan (2009) found the LHS too difficult for developing influence diagnostics, and it is also cumbersome to work with for estimating MSY. In their extensive review of segmented regression, Seber and Wild (1989) examined several models accommodating a smooth transition around the join-point (or breakpoint) between two segments. In particular, the bent-hyperbola model (Watts and Bacon, 1984) seems to serve the purpose well.

The general form of the bent hyperbola (in S–RR terms), as a transition between two linear segments ("left" and "right"), is

$$R = \phi(S) = \beta_0 + \beta_1(S - S^*) + \beta_2\sqrt{(S - S^*)^2 + \gamma^2/4}, \tag{3}$$

where S^* is the biomass breakpoint, and γ is a measure of the radius of curvature near the breakpoint; as γ approaches zero, the curve has a sharp bend similar to the HS. If the left segment has a slope θ_1, and the right segment a slope θ_2, the following applies for β_1 and β_2:

$$\beta_1 = (\theta_1 + \theta_2)/2,$$
$$\beta_2 = (\theta_2 - \theta_1)/2.$$

As we want to mimic the HS, we have $\theta_2 = 0$ (the right segment horizontal), so $\beta = \beta_1 = -\beta_2 = \theta_1/2$. Moreover, it is desirable that the curve passes through the origin (no recruit if no parents). This leads to the Watts–Bacon bent hyperbola:

$$R = \phi(S) = \beta\{S + \sqrt{S^{*2} + \gamma^2/4} - \sqrt{(S - S^*)^2 + \gamma^2/4}\}. \tag{4}$$

This model has the satisfactory property that it curves inside the corner at the intersection of the asymptotes, whose slopes are

$$\frac{d\phi}{dS} = \begin{cases} 2\beta, & S \to 0 \\ 0, & S \to \infty \end{cases}.$$

When fitting the model to data, the parameters to estimate are β, S^*, and γ. The R^* plateau is found by applying Equation (4) to large values of S; it can also be obtained analytically as $R^* = \beta(S^* + \sqrt{S^{*2} + \gamma^2/4})$. However, Seber and Wild (1989) pointed out that there are generally too few data around the breakpoint to describe the transition well, and γ is likely to be estimated with poor precision. Moreover, when γ is a free parameter, the sum-of-squares surface is poorly conditioned, leading to problems with minimization. Hence, they advised to hold it fixed. Watts and Bacon (1984) further noted that their estimates were insensitive to the prior choice of γ, and Toms and Lesperance (2003) observed that a range of values of γ (including a sharp model) was equally plausible. Our experience is also that minimization algorithms often do not move the starting value of γ, and the returned hessian has values of zero in rows and columns for γ. Hence, we followed the advice of Seber and Wild (1989) in our examples and searched β and S^* for fixed trial values of γ. Because we wanted to mimic assessments with a HS, we chose small values for γ (0.01, 0.1, or 0.5), but, as can be seen in an example, larger values did not change the estimates of S^* and R^*.

F_{MSY} with a continuous HS

It is easy to cast Equation (4) into Equation (1) and solve

$$S_e = \lambda_F \beta\left\{S_e + \sqrt{S^{*2} + \gamma^2/4} - \sqrt{(S_e - S^*)^2 + \gamma^2/4}\right\}$$

for S_e, knowing the SSB-per-recruit λ_F for a given F. This leads to a simple quadratic equation, the non-trivial (positive) solution of which is

$$S_e = \frac{2K/(\lambda\beta) - 2S^* - 2K}{1/(\lambda^2\beta^2) - 2/(\lambda\beta)},$$

where $K = \sqrt{S^{*2} + \gamma^2/4}$, and the F subscript to λ is dropped. When divided by λ_F, this root gives the level value R^*, which corresponds to R_e. However, above some threshold value of fishing mortality, the slope of the replacement line exceeds the slope of the ascending S–RR segment (which is $\theta_1 = 2\beta$). As with other S–RR, this indicates that the replacement line is too steep and has no (positive) intercept with the stock–recruit curve, such that the stock is "heading for its graveyard" (Beverton, 2002). A self-explanatory notation for this threshold is F_{crash}. Hence, all F values (or factors) such that $1/\lambda_F > 2\beta$ are earmarked as non-viable for a long-term yield.

Examples

Fitting a bent-hyperbola S–RR to simulated data

A set of stock-and-recruitment data emulating a HS was generated, 30 values of spawner biomass S being obtained by randomly sampling numbers in the range 1–2000. The corresponding recruits were estimated from Equation (2), assuming a threshold S^* of 750 and an initial slope a of 1.25. A lognormal noise (as commonly assumed when dealing with recruitment) with a CV of 0.3 was then added. The bent hyperbola [Equation (4)] was fitted to the data with a fixed γ of 0.5. For comparison, a HS was also fitted with the profiling method suggested by Barrowman and Myers (2000), yielding parameters that differed only marginally from those provided by the grid-search or Julious methods used by ICES (2002). The plot of both fits is shown in Figure 1.

The bent hyperbola closely matches the HS, both for the change-point S^* and the R^* plateau, although no parameter was set to force this in any way (note that, because of added noise, both fits give estimates that differ from the initial specification: 821 instead of 750 for S^*, and 971.6 rather than 937.5 for R^*). As a small γ was used, even the shape near the breakpoint is indistinct. We also checked the lack of sensitivity to the choice of γ. Figure 1 shows that there is no visible difference in the fits when γ is extended from 0.5 to 1 or 10 and that one needs to drag γ to extreme values, >100, to change the pattern appreciably.

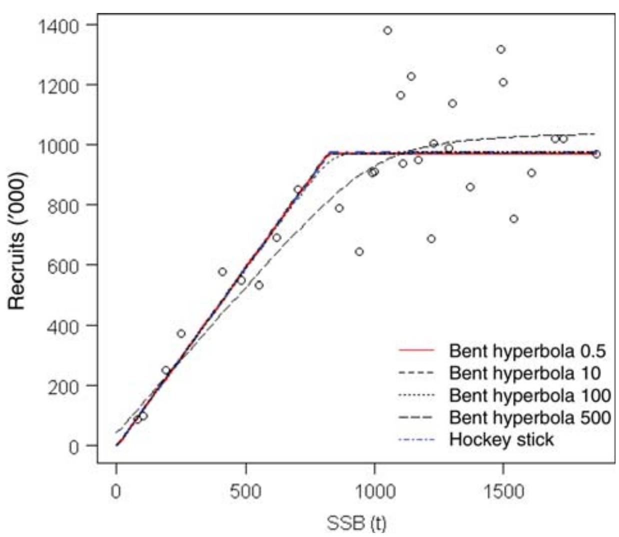

Figure 1. Plot of the bent-hyperbola S–RR with γ fixed at 0.5 (solid, red), and of three alternative assumptions for the curvature parameter γ, on artificial data (open circles); a HS fit to the same data is also plotted (dashed, blue).

Estimating F_{MSY} for Baltic cod

A case study with estimation of F_{MSY} is hard to find in ICES reports, so we resorted to the FAO publication by Lassen and Medley (2000), with its annexed spreadsheets where a worked example is presented. The example is based on eastern Baltic cod (*Gadus morhua*), with stock-and-recruitment (at age 2) data for the 1966–1994 year classes. Input data for per-recruit analyses (natural mortality M, selection pattern s, weights W in the catch and the stock, maturity O, all by age) are also provided. An additional example for North Sea cod is provided in our Supplementary material.

The bent hyperbola was fitted to the stock–recruit data with different trial γ; there was no obvious difference in parameter estimates, and we kept $\gamma = 1.0$. Again, the fitted curve is fully superimposed over the HS curve. The example in Lassen and Medley (2000) assumes a Beverton and Holt S–RR, and we also fitted one to the data. Figure 2 illustrates one of the contentions of Barrowman and Myers (2000) that the Beverton–Holt S–RR tends to give higher recruitment at medium or high biomasses, leading to overoptimistic long-term yields.

A per-recruit analysis provides yield-per-recruit and spawning-biomass-per-recruit for a range of values of F. We then used the procedure described earlier to estimate the equilibrium biomass S_e and the equilibrium recruitment R_e, and hence the equilibrium yield for each F. F_{MSY} is obtained for an F of 0.477 (mean over ages 4–7). For this value of F, the replacement line cuts the S–RR curve to the right of the threshold, where recruitment is constant at R^*; therefore, this is the same F as for F_{max} on a yield-per-recruit curve. However, this F is only 73% of F_{crash} (0.65), where there is no intersection between the replacement line and the S–RR curve. By comparison, with a Beverton–Holt S–RR, Lassen and Medley (2000) found a smaller F_{MSY} of 0.317 and a much larger F_{crash} of 1.36. The

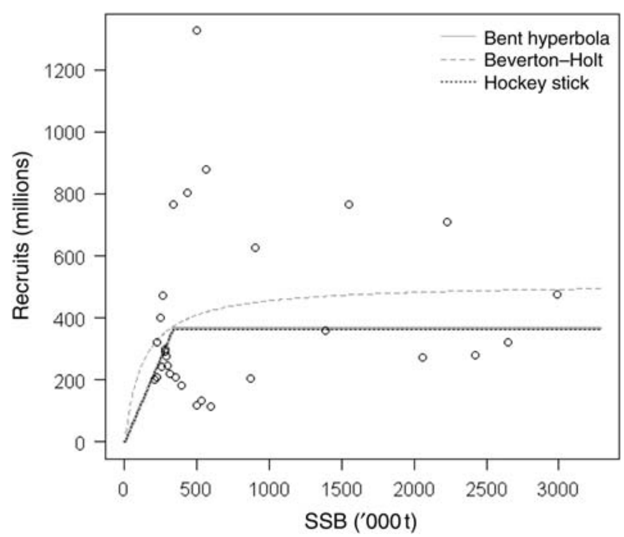

Figure 2. Comparison of three stock–recruit models for Baltic cod. Data pairs (year classes 1966–1994) are shown as open circles.

equilibrium yield for the two S–RR is shown in Figure 3. Note the abrupt drop at F_{crash} with the bent hyperbola, whereas yield decreases gradually with a Beverton–Holt S–RR. An advantage of the F_{MSY} procedure is the substantiation that yield can be annihilated when a threshold in fishing mortality is exceeded, whereas a yield-per-recruit analysis gives no such indication.

As the maximum is on a relatively flat portion of the yield curve and F_{MSY} is close to F_{crash}, it may be of interest to consider the uncertainty in estimating F_{MSY}. The stock-and-recruitment data are the results of catch-at-age analyses using models of varying complexity, subject to intricate effects of errors in reported catch, discards, assumed natural mortality, etc. The estimates of F and selection pattern used in per-recruit analyses are also produced by the same channel. Without access to the original data, it is impossible to incorporate all possible sources of noise. A simple alternative is the jackknife approach, where the bent-hyperbola fit and F_{MSY} estimation are repeated with each stock–recruit pair dropped in turn. In this instance, the jackknife variance is extremely small, and F_{MSY} differences occur only at the sixth decimal place. There are indications that dropping the pairs with high (or low) recruitment tends to reduce (or raise) the estimate of F_{MSY}, but the absolute differences are infinitesimal.

Discussion

Although the concept of MSY has been debated intensely within the scientific community, owing to the various conceptual and technical issues, political authorities have recently restated their attachment to this long-standing management objective. Their will is that fishing mortality should be brought to and maintained near F_{MSY}. Therefore, many European Union recovery or management plans include an F_{MSY} policy, and scientists are requested by managers to evaluate the implications of fishing at F_{MSY} relative to alternative policy options. At a minimum, this implies that scientists have the capacity to estimate F_{MSY}. With the class of models used by ICES, the procedure requires the specification of a trustworthy S–RR.

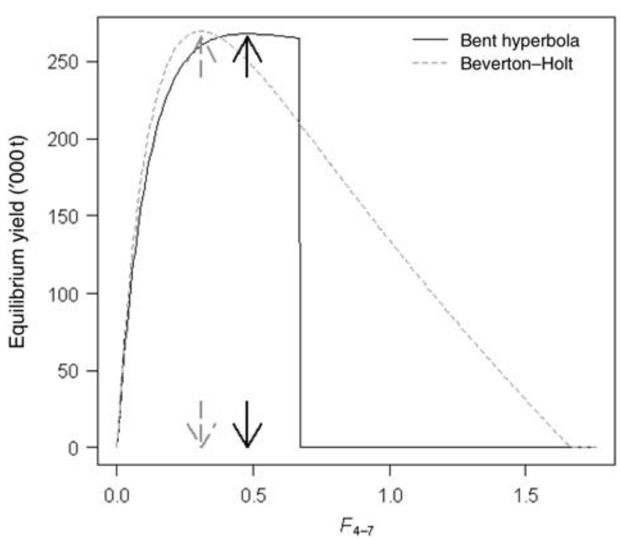

Figure 3. Comparison of equilibrium yields and positions of F_{MSY} (arrows) for a bent hyperbola and a Beverton and Holt S–RR for Baltic cod.

Here, the procedure to estimate the genuine F_{MSY} established for conventional S–RR (e.g. Shepherd, 1982; Quinn and Deriso, 1999) has been extended to an additional S–RR, the HS. This segmented form of S–RR is often selected when the evidence for more elaborate functions is weak, given the data at hand, and goodness-of-fit statistics are at times better than those of competing models (Barrowman and Myers, 2000). Theoretical considerations indicate that HS parameters (the breakpoint and the level recruitment) are "design-robust", i.e. less sensitive to the addition or deletion of S–R pairs (Cadigan, 2006), although some practitioners argue that adding new observations near the origin or far from the breakpoint significantly change the parameters (ICES, 2007). Whether such change in the S–RR parameters has a significant impact on the estimate of F_{MSY} needs to be checked in each specific case.

However, if the choice is made to assume a HS S–RR, the original formulation is not convenient for analytical estimates of MSY, and there is a need for a continuous, differentiable alternative. Among the several functions quoted in the literature, the bent-hyperbola variant proposed by Watts and Bacon (1984) proved most suitable. Although it comes from a different lineage than the methods normally used to fit a HS, the estimated curve is remarkably similar to the HS curve, except that, as desired, it is continuous locally about the breakpoint. The shape of this continuity is determined by a parameter that can be varied over a broad range without significant changes in the other parameters of interest, viz. the breakpoint S^* and the level recruitment R^* when SSB is above the breakpoint. Previously, resorting to smooth variants of the HS was found to be a requirement for computing confidence intervals based on the likelihood (Cadigan and Healey, 2004). The bent hyperbola probed here is one of a family of functions studied by Toms and Lesperance (2003), and it is straightforward to carry out the same inference studies with various error distributions, as done by those authors. It is also trivial to perform leave-one-out simulations to check the robustness of parameters when observations are added or deleted and to include the plot in standard diagnostics, as done by ICES (2002).

The focus here was on calculating F_{MSY} deterministically, such that the estimate can be carried into catch forecasts. There are indications that ICES would rather advise on ranges for F_{MSY}, taking account of variability around the stock–recruit curve, in growth, natural mortality, selection pattern, etc. Nevertheless, the essential steps in the calculation will remain as shown here, and likewise for management plan simulations, where computations are done over many replicates.

To estimate MSY reference points, one needs to consider an equilibrium recruitment located at the intersection of the replacement line under a given F with the S–RR curve. When the S–RR is of the HS type, the intersection for low-to-moderate Fs takes place where recruitment is constant at the R^* plateau, so F_{MSY} has the same value as F_{max}, the value for which the yield-per-recruit is maximized (a distinct concept). When $F > F_{crash}$, i.e. where both curves have no intersection, the equilibrium recruitment and yield drop suddenly to zero. There may be a problem with stocks exhibiting a flat-topped, yield-per-recruit curve with a maximum for high F values, possibly above F_{crash}, depending on the steepness of the replacement line (low SSB per recruit). In that case, F_{MSY} will be well beneath F_{max}. This shows that F_{max} can be a risky proxy for F_{MSY}, because it does not account for the stock–recruitment process and may at times be dangerously close to F_{crash} (Punt and Smith, 2001). ICES should not recommend its use in lieu of F_{MSY} (ICES, 2009a).

Supplementary material

Supplementary material is available at *ICESJMS* online for a North Sea cod example. Two references cited only in the Supplementary Material are included below in the list of references, for completeness.

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