

Statistics of wave crests from second order irregular wave 3D models

Marc Prevosto

IFREMER Centre de Brest

B.P. 70, F-29280 Plouzané, France

Marc.Prevosto@ifremer.fr

WWW home page: <http://www.ifremer.fr/metocean/>

Abstract. The statistics of the elevation and kinematics of waves in real seas are very rarely accessible from *in-situ* measurements which induced very high costs. An alternative to the actual waves observation is to derive from the spectral climatology of waves, completed with other environmental data like the wind and the current, the statistics of the individual waves. But this requires accurate models of irregular gravity waves which take into account all the main nonlinearities and interactions with the local wind, current and bathymetry. In a first step, methodologies have been based on second order irregular wave 3D models and have supplied the engineers with new better accurate models of statistics of wave crests. These models do not include yet the complex interactions with wind and current but participate to the improvement of the tools for the design of offshore structures.

1 *In-situ* Measurements

The statistics of the elevation and kinematics of waves in real seas have been greatly based for specific site studies on *in-situ* measurements (North-Sea and Gulf of Mexico oil fields). The incomparable great quality of a measurement is that it includes all the physical phenomena, but unfortunately also those which corrupt the actual observation of waves (mooring behavior and transfer function for buoys, fouling effect for plunged or underwater probes, sea foam or spray effect). To this list will be added the problems of spatial integration, calibration and data transformation and transmission. So it becomes difficult to clean the measurements without degrading the extreme or unexpected events. Moreover the wave instruments furnish point measurements and so the instrumentation might be very expensive and long to build accurate statistics, making cost and duration time not always compatible with the constraints of the project on the site. Apart for some very rich data base, measurements will be used to analyze typical situations and to validate or invalidate models.

So the question is: *Is it reasonably possible to build accurate statistics of wave kinematics from wave measurements?* The answer is obviously **No!** Apart for some very extensive data base (e.g. North-Sea and Gulf of Mexico oil fields).

The alternative issue is then: *Is it reasonably possible to build accurate statistics of wave kinematics from wave models?* This is what attempt to answer a lot of works this last ten years mainly in using nonlinear irregular wave 3D models.

2 Power Spectra versus Wave by Wave

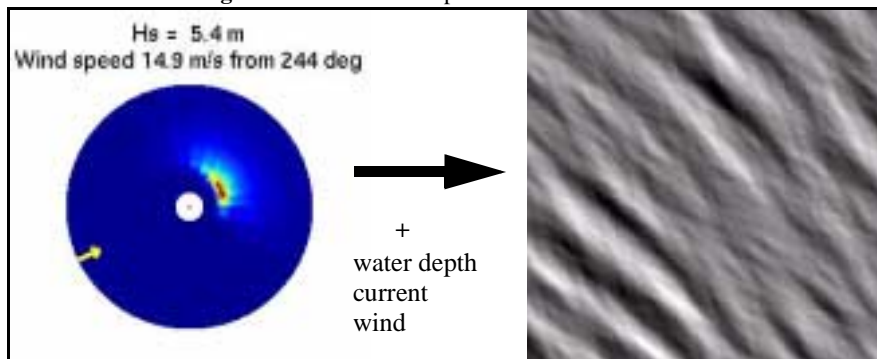
More and more information on waves are restricted to information on energy. The hindcast models use better wind fields and assimilate larger amount of data (e.g. satellite). They use better models of generation, interaction and dissipation and profit by the always increasing power of the computers. The satellites, too, furnish spectral information with the SAR (directional spectrum) or the altimeters (Hs).

The so-called “Wave forecast” of the Meteorological Offices consists in the forecast of sea states (Hs, main direction or directional spectrum) and the step to forecast the corresponding stochastic information on the wave kinematics, is a giant step if we know that we have to collect information and to input in the stochastic models local currents, winds and bathymetry and to take into account complex phenomena, nonlinearities and breaking effects. To take such a giant step, the addition of small steps will be necessary, some of them have been already taken that we describe hereafter.

The advantages of working with the spectral information is that this information i) is available all over the world (limited to the grid of the models or to the time-space sampling of the satellite tracks), ii) has been collected or computed for several years (up to 40 years for the hindcast models and 15 years for the satellites), iii) is available in forecast problems thanks to forecast wind fields as input of wave hindcast models.

The difficult passage from spectral to wave by wave information is illustrated in figure 1.

Fig. 1. From directional spectra to wave kinematics



3 Methodologies for Statistics

The methodologies to furnish statistics of waves inside a sea state starting from spectral information are of different kinds. They can be based on Monte Carlo techniques and development of simulators (Forristall [3,10], Prevosto [8,10]), or derived from theoretical considerations: Transformed Gaussian process method (Rychlik [11]), First Order Reliability Method (FORM) (Tromans [13]).

Starting from measurements or from simulation or theoretical methodologies, simplified parameterized models based on a fitting procedure have been proposed as better practical tools for the engineers.

In any case, independently of the methodology, the answers will differentiate from the model of irregular gravity waves taken as starting point.

3.1 Linear Model

The simplest linear model of superposition of Airy waves used the directional spectral density $S(\theta, f)$ as statistical information on the variance of the amplitudes of the components.

$$\eta_1(t) = \sum_{\theta, f} b(\theta, f) \sin(2\pi f t) + c(\theta, f) \cos(2\pi f t) = \sum_{\theta, f} a(\theta, f) \sin(2\pi f t + \phi(\theta, f)) \quad (1)$$

with b and c Gaussian random variables defined by

$$\mathbf{IE}(b(\theta, f)^2) = \mathbf{IE}(c(\theta, f)^2) = S(\theta, f) d\theta df \text{ and } \mathbf{IE}(b(\theta_i, f_j) c(\theta_k, f_l)) = 0 \quad (2)$$

This model furnishes a Rayleigh law as the law of the crest heights. (3)

$$P(C > c) = \exp\left(-8 \frac{c^2}{H_s^2}\right) \text{ with } H_s = 4 \sqrt{\iint S(\theta, f) d\theta df} \quad (4)$$

3.2 Non Linear Models

Wave height considered as the crest-trough amplitude (and this definition could be extended to other parameters, e.g. crest-trough pressure, crest-trough velocity as soon as kinematics is studied under the mean water level) are influenced by the steepness nonlinearity at one higher order of magnitude than the crest or trough amplitudes. This explain the good fitting and quality of models of wave heights based on the linear assumption.

But more complicated models have to be considered to take into account the strong effect of the nonlinearities on the crest amplitudes (or other amplitude of the kinematics), e.g. the hybrid model (Zhang [15]), the Creamer-transformation (Creamer [1]) or the Stokes 5th order correction (Dawson [2]). But as an intermediate way, which take into account the wave spreading, irregular 2nd order 3D models have been extensively used and validated for the last years.

3.3 Stokes 2nd Order Based Models

2nd Order Directional - 3D Wave Model. The 2nd order Stokes expansion based on the linear part (Eq. 1) is

$$\begin{aligned} \eta_2(t) = & \sum_{\theta_i, f_j, \theta_k, f_l} a(\theta_i, f_j) a(\theta_k, f_l) T^D(\theta_i, f_j, \theta_k, f_l) \cos(2\pi(f_j - f_l)t + (\phi(\theta_i, f_j) - \phi(\theta_k, f_l))) \\ & + \sum_{\theta_i, f_j, \theta_k, f_l} a(\theta_i, f_j) a(\theta_k, f_l) T^S(\theta_i, f_j, \theta_k, f_l) \cos(2\pi(f_j + f_l)t + (\phi(\theta_i, f_j) + \phi(\theta_k, f_l))) \quad (5) \\ & - c_{\eta_2} \end{aligned}$$

where c_{η_2} is a constant to ensure that $\mathbf{IE}(\eta_2(t)) = 0$.

The two 2nd order transfer functions T^S and T^D of course depend of the water depth. Their expressions are given in appendix 1.

2nd Order Uni-Directional - 2D. If now we consider a uni-directional wave train in which all the components propagate in the same direction, we obtain, of course, the same linear part of the elevation

$$\eta_1(t) = \sum_{\theta, f} a(\theta, f) \sin(2\pi f t + \phi(\theta, f)) = \sum_f a_u(f) \sin(2\pi f t + \phi_u(f)) \quad (6)$$

but a different second order part in applying Eq. 5 with $a(\theta, f) = a_u(f)$, $\phi(\theta, f) = \phi_u(f)$ and $\theta_i = \theta_k = 0$, and calculating $T_u^S(f_j, f_l)$, $T_u^D(f_j, f_l)$.

3.4 Crest Height Probability Distribution

Mainly focussed on the aim to produce simple parametric models corresponding to uni-directional or directional sea states and to infinite to intermediate water depths, some authors proposed and fitted crest height probability distribution models based on perturbations of the laws of the linear case. Some used measurements, others the Stokes 2nd order irregular waves models.

Jahns & Wheeler. This model is based on a nonlinear transformation of a Rayleigh law, where the transformation is dependent of the crest height normalized by water depth (Jahns & Wheeler [5]). This model has been fitted later from measurements (Haring & Heideman [4]). It appears clearly wrong in infinite depth where it tends to the Rayleigh law. The fitting used wave staff measurements in the Gulf of Mexico and Waverider measurements in the North Sea.

$$P(C > c | H_s; h) = \exp\left(-8 \frac{c^2}{H_s^2} \left(1 - 4.37 \frac{c}{h} \left(0.57 - \frac{c}{h}\right)\right)\right) \quad (7)$$

Derived Narrowband Models. Some other models were derived from a narrowband model of the 2D second order irregular waves model. This model obtained from Eqs 1 & 5, in the 2D case, is valid if the spectral density is sufficiently narrow to consider the 2nd order transfer functions as constant. In this case,

$T_u^D(f_j, f_l)$ (resp. $T_u^S(f_j, f_l)$) are considered constant and equal to $T_{nb}^D(f_m)$, (resp. $T_{nb}^S(f_m)$), with

$$T_{nb}^D(f_m) = \lim_{\substack{f_j \rightarrow f_m \\ f_l \rightarrow f_m}} T_u^D(f_j, f_l) \quad \text{and} \quad T_{nb}^S(f_m) = \lim_{\substack{f_j \rightarrow f_m \\ f_l \rightarrow f_m}} T_u^S(f_j, f_l) \quad (8)$$

with f_m a mean frequency to be defined. This gives for the second order part

$$\begin{aligned}
\eta_2(t) = & T_{nb}^D(f_m) \sum a_u(f_j) a_u(f_l) \cos(2\pi(f_j - f_l) + (\phi_u(f_j) - \phi_u(f_l))) \\
& + T_{nb}^S(f_m) \sum_{f_j=f_l} a_u(f_j) a_u(f_l) \cos(2\pi(f_j + f_l) + (\phi_u(f_j) + \phi_u(f_l))) \\
& - T_{nb}^D(f_m) \sum_{f_j} a_u^2(f_j)
\end{aligned} \tag{9}$$

if $\eta_1(t)$ is considered as a product of an amplitude and a phase time function, $\eta_1(t) = A(t)\cos(\Omega(t))$, where the amplitude and instantaneous frequency are slowly varying, the unidirectional narrowband second order part becomes

$$\begin{aligned}
\eta(t) = & A(t)\cos(\Omega(t)) \\
& + T_{nb}^D(f_m)A^2(t) + T_{nb}^S(f_m)A^2(t)\cos(2\Omega(t)) \\
& - \frac{1}{8}H_s^2T_{nb}^D(f_m)
\end{aligned} \tag{10}$$

The formulas for T_{nb}^D and T_{nb}^S are given in appendix 2.

If we consider that the envelope varies sufficiently slowly, the crest occurs at instant t_c when. $\Omega(t_c) = 0$. Then the crest height given by the linear part is $A(t_c)$, and the crest height at second order is

$$A_{nonlin}(t_c) = A(t_c) + (T_{nb}^D(f_m) + T_{nb}^S(f_m))A^2(t_c) - T_{nb}^D(f_m)\frac{H_s^2}{8} \tag{11}$$

which links linear to nonlinear crest heights by a quadratic transformation:

$$C = C_{lin} + (T_{nb}^D(f_m) + T_{nb}^S(f_m))C_{lin}^2 - T_{nb}^D(f_m)\frac{H_s^2}{8} \tag{12}$$

Tayfun [12], Tung and Huang [14], Kriebel and Dawson (1991) [6], Kriebel and Dawson (1993) [12] and Prevosto *et al.* [9] proposed models based on such a non-linear quadratic relation:

$$C = C_{lin} + \alpha(f_m; h)C_{lin}^2 + \beta = Q(C_{lin}) \tag{13}$$

and on the Rayleigh law for the distribution of the linear crests:

$$P(C_{lin} > c | H_s) = \exp\left(-8\frac{c^2}{H_s^2}\right) \tag{14}$$

So, in a classical way, the distribution of the nonlinear crests is obtained by applying the inverse nonlinear transformation.

$$P(C > c | H_s; f_m, h) = \exp\left(-8\frac{(Q^{-1}(c))^2}{H_s^2}\right) \tag{15}$$

The only solution of the inverse transformation is

$$C_{lin} = Q^{-1}(C) = \frac{-1 + \sqrt{1 + 4\alpha(C - \beta)}}{2\alpha} \tag{16}$$

giving

$$P(C > c | H_s; f_m, h) = \exp\left(-8\frac{\left(\frac{-1 + \sqrt{1 + 4\alpha(c - \beta)}}{2\alpha}\right)^2}{H_s^2}\right) \tag{17}$$

The differences between the parametric models proposed by a number of authors come from different choices of $\alpha(f_m; h)$ and β , and different approximations of $Q^{-1}(C)$. All the previous authors apart Prevosto *et al.* [9] take β equal to zero and coefficient of the transformation from second order regular Stokes wave. But unfortunately in finite water depth the irregular narrowband models do not tend to the regular model (due to the difference terms), making the Kriebel and Dawson finite depth model not an exact one (Compare Eq. 20 to the sum of Eq. 36 and Eq. 37). Tung and Huang [14] made an error by taking into account in infinite water depth a low frequency part which in fact does not exist (Eq. 36).

Kriebel and Dawson. The Kriebel and Dawson model is based on the second order regular Stokes wave model in infinite or finite depth, giving

$$C = C_{lin} + \frac{1}{2H_s} C_{lin}^2 \rightarrow C_{lin} = \left(-1 + \sqrt{1 + 2\frac{R}{H_s}C} \right) \frac{H_s}{R} \quad (18)$$

$$\text{with } \begin{cases} R = kH_s f_2(kd) \\ k \leftarrow T_m = 0.95T_p \end{cases} \quad (19)$$

$$\text{and } f_2(kd) = \frac{\cosh kd(2 + \cosh 2kd)}{2\sinh^3 kd} - \frac{1}{\sinh 2kd} \quad (20)$$

Kriebel and Dawson approximated the inverse transformation $Q^{-1}(C)$, first [6] at second order and later [7] with a corrected third order expansion. This induces a problem in the crest distribution when the steepness is strong. These simplifications are not necessary as we know an analytic form of the inverse transformation (Eq. 16).

In infinite depth the exact Kriebel and Dawson model and the Tayfun model are the same. A difference could exist which comes from the definition of T_m (Eq. 19).

$$P(C > c | (H_s, T_p)) = \exp\left(-\frac{8}{H_s^2 k^2} (-1 + \sqrt{1 + 2kc})^2\right) \quad (21)$$

The same technique is used in (Dawson [2]) with a 5th order regular expansion. These models, though based in their principle on narrowband assumptions, do not use an exact narrowband Stokes expansion. This induces errors in the models, apart in infinite depth where harmonic and narrowband expansion are the same.

3.5 New Models

Two new models have been recently proposed and take into account the 3D structure of the waves.

Forristall Model. It is based on a perturbed Weibull law with the two parameters written as steepness and Ursell number polynomials (Forristall [3]). Starting from simulations based on a synthetic directional spectrum data base and different water depths, two different sets of coefficients of the polynomials were fitted from 2D and 3D simulations.

$$P(C > c) = \exp\left(-\left(\frac{c}{\alpha H_s}\right)^\beta\right) \quad (22)$$

$$\text{with } \alpha = \alpha_1 + \alpha_2 S_1 + \alpha_3 U_r, \beta = \beta_1 - \beta_2 S_1 - \beta_3 U_r + \beta_4 U_r^2, \quad (23)$$

$$\text{the steepness } S_1 = \frac{2\pi H_s}{g T_{01}^2} \text{ and the Ursell number } U_r = \frac{H_s}{k_{01}^2 d^3}.$$

The fit on 2D simulations gave

$$\alpha = 1/\sqrt{8} + 0.2892 S_1 + 0.1060 U_r \quad (24)$$

$$\beta = 2 - 2.1597 S_1 + 0.0968 U_r^2$$

The fit on 3D simulations gave

$$\alpha = 1/\sqrt{8} + 0.2568 S_1 + 0.0800 U_r \quad (25)$$

$$\beta = 2 - 1.7912 S_1 - 0.5302 U_r + 0.284 U_r^2$$

The advantage of this model is its simplicity, but it does not take into account variations in the directional spreading.

Prevosto Model. It is based on a nonlinear transformation of a Rayleigh law, where the transformation is based on the narrowband Stokes expansion (Prevosto [10]). The two parameters H_s and mean wavenumber are perturbed to take into account the spectral bandwidth, the directional spreading and the water depth in equations 12-14. It has a unique expression in 2D and 3D case.

$$\tilde{H}_s = \alpha_{H_s} H_s, \tilde{f}_m = \alpha_{f_m} f_m \quad (26)$$

In looking at different directional spectrum climatologies and different water depths, the α_{H_s} and α_{f_m} formulations have been determined from simulations and theoretical considerations to be:

$$\alpha_{H_s} = 1 - \frac{1}{2} (\tanh(kd) - 0.9) \sqrt{\frac{2}{1+s}} \quad (27)$$

where s is the power of the equivalent \cos^{2s} directional distribution at the peak frequency, and

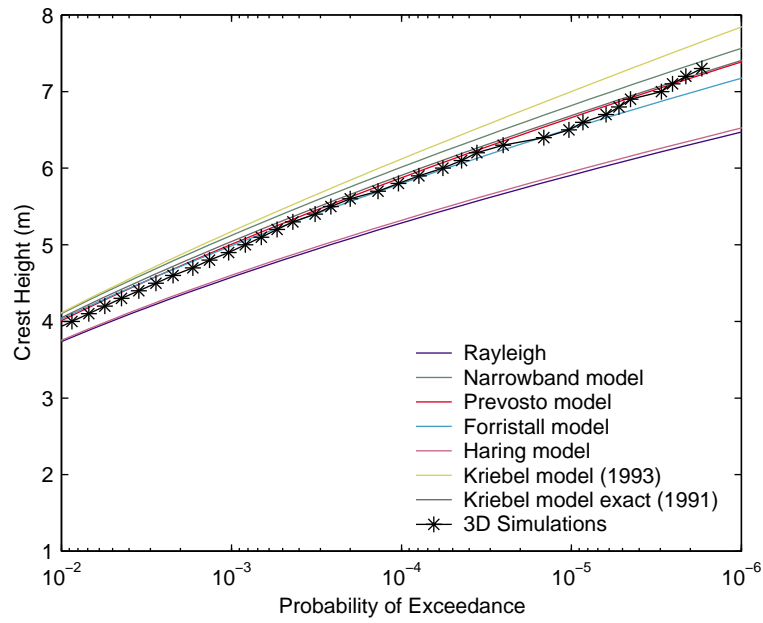
$$\alpha_{f_m} = \frac{1}{1.23} \text{ with } f_m = \frac{1}{T_{02}} \quad (28)$$

The formulation of α_{H_s} has been chosen to take into account the fact that the effect of the directional spreading on the crest heights is opposite in deep and shallow water (see [8]). This model has the advantage of furnishing a unique expression both the 2D and 3D cases, and so can be adapted to all intermediate situations.

3.6 Comparison of the Models

These models have been compared to the empirical distribution of crest heights calculated from 1000 hours of simulations (3D second order irregular waves model) of a sea-state with parameters ($H_s=5\text{m}$, $T_{02}=7\text{s}$, $s=11$). Three different water depths have been used (1000m, 30m, 20m). It is clear that in all the cases (Figs. 2-4), Forristall and Prevosto models give very good results. In the deep water case the Haring (jahns & Wheeler) model is close to Rayleigh and in shallow water the Kriebel models are not at all accurate.

Fig. 2. Water depth 1000 meters



4 Validity of the 2nd Order Models in Extreme Situations

The use of 2nd order models has the advantage to work with simple wave models. If these models are used to calculate design crest heights, their validity has to be proved before using such extreme values.

Fig. 3. Water depth 30 meters

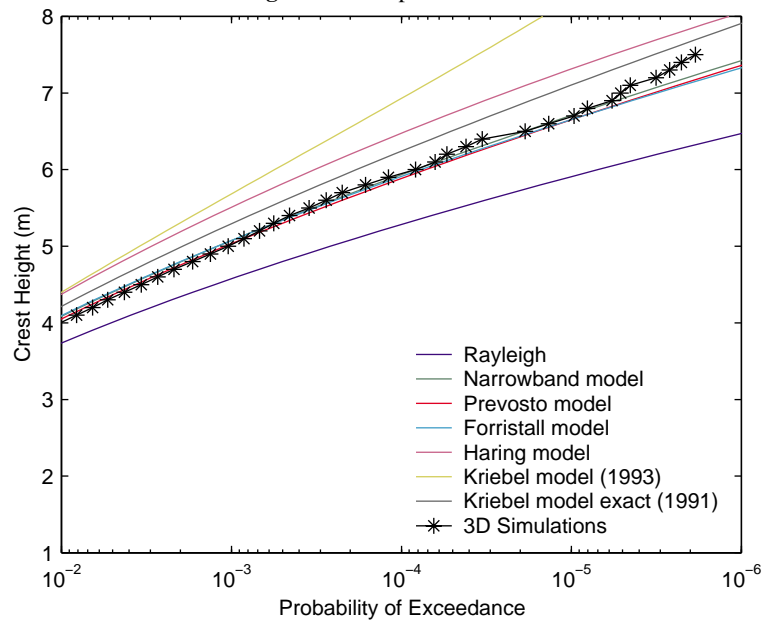
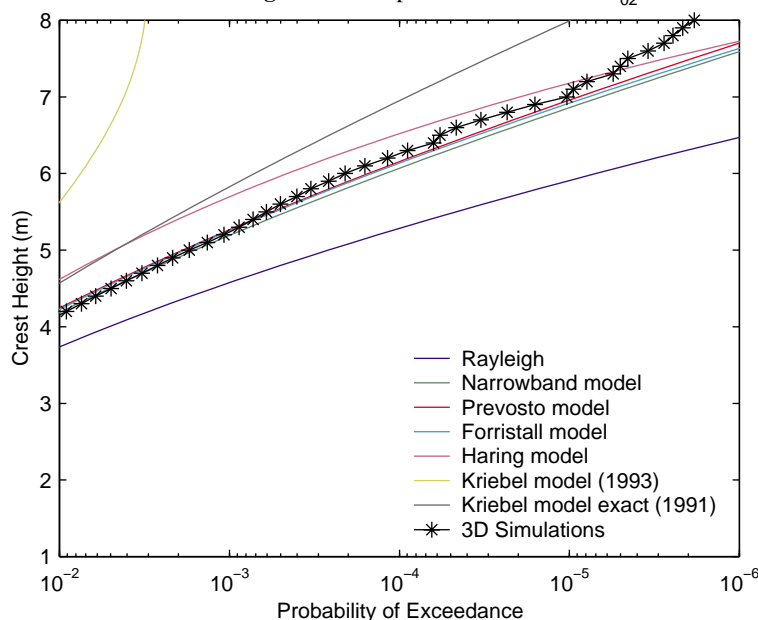


Fig. 4. Water depth 20 meters



The biggest crest encountered during the 1000 hours (Fig. 5, red curve), in the 1000 meters water depth case, has a wave height of 12 meters, a crest height of 7.4 meters (1.5 times the H_s), a wave period of 8.5 sec and a crest duration of 4 sec. This wave has a crest shape very close to the breaking limit. In this case the difference between the 2D and 3D models is very small compared to the modification of the shape of the wave due to the 2nd order nonlinearity. If now we consider an harmonic wave with a 5th order expansion giving the same crest height, wave height and crest duration (Fig. 6) we observe that the 2nd order expansion for this very extreme wave is not so far from the higher expansion and that the main improvement in the model is from linear to second order. This, of course, does not validate the distributions based on the 2nd order irregular waves models, but shows that accurate distribution models like the two Forristall and Prevosto models permits in a first step to greatly improve the tools for the design of offshore structures.

A 3D view of this biggest crest is given in figure 7, which shows the complexity of the shape and of the slopes of such a wave and so the difficulties to define it as a dangerous or not dangerous wave.

5 Conclusion

As an alternative to the actual waves observation, the use of the spectral information combined with models of irregular gravity waves has permitted to supply the engineers with new better accurate distributions of wave crests. These distributions have been fitted starting from 3D second order irregular waves models. If partly validated for the crest heights, this methodology will not be enough accurate for other parameters of the crest kinematics which ask for higher order expansion. Moreover, the introduction of breaking, local wind and current will introduce certainly modifications in the probability of occurrence of extreme kinematics. But at

the moment, to take into account in the irregular wave models local wind and current is a big issue not yet solved, which will be the next step for the improvement of the design tools and to progress in the maritime risk assessment.

Fig. 5. The biggest crest

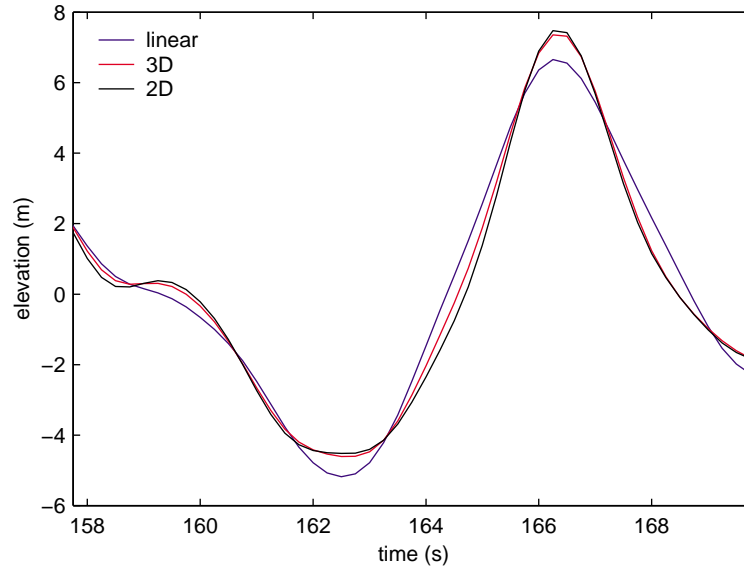


Fig. 6. The equivalent regular wave

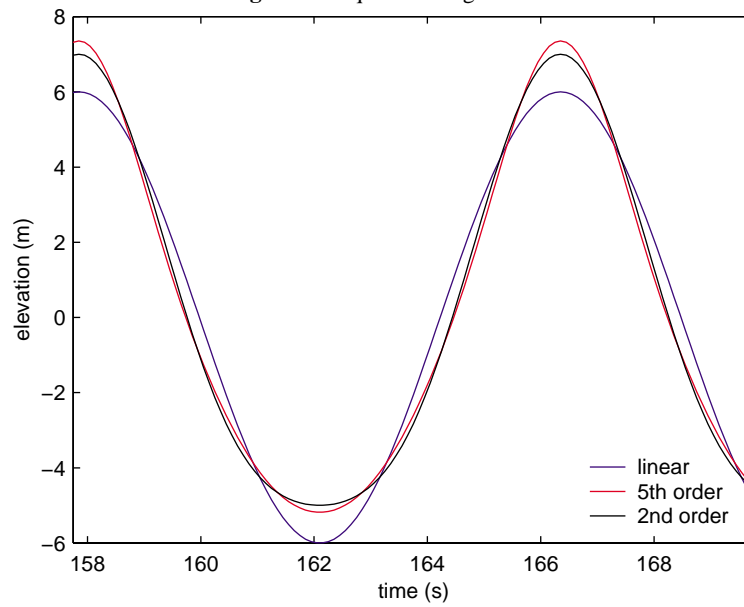


Fig. 7. 3D view of the biggest crest



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Appendix 1: Second Order Transfer Functions:

$$T(\theta_i, f_j, \theta_k, f_l) = \frac{1}{2g} \left(2(\omega_j + \omega_l) D(\vec{k}_i, \vec{k}_k) + \omega_j \omega_l + \omega_j^2 + \omega_l^2 - \frac{g^2 \vec{k}_i \bullet \vec{k}_k}{\omega_j \omega_l} \right) \quad (29)$$

with

$$\vec{k}_i = k_j \vec{d}_i, \quad k_j = |\vec{k}_i|, \quad \vec{d}_i = \cos \theta_i \vec{x} + \sin \theta_i \vec{y}, \quad (2\pi f_j)^2 = \omega_j^2 = g k_j \tanh k_j h \quad (30)$$

$$\vec{k}_k = k_l \vec{d}_k, \quad k_l = |\vec{k}_k|, \quad \vec{d}_k = \cos \theta_k \vec{x} + \sin \theta_k \vec{y}, \quad (2\pi f_l)^2 = \omega_l^2 = g k_l \tanh k_l h \quad (31)$$

with h the water depth and

$$D(\vec{k}_i, \vec{k}_k) = \frac{2(\omega_j + \omega_l)(g^2 \vec{k}_i \bullet \vec{k}_k - \omega_j^2 \omega_l^2) + g^2(k_j^2 \omega_l + k_l^2 \omega_j) - \omega_j \omega_l (\omega_j^3 + \omega_l^3)}{2\omega_j \omega_l ((\omega_j + \omega_l)^2 - g |\vec{k}_i + \vec{k}_k| \tanh |\vec{k}_i + \vec{k}_k| h)} \quad (32)$$

$$D(k\vec{d}_i, \omega, k\vec{d}_k, -\omega) = 0 \quad (33)$$

Appendix 2: Narrow-Band Non Linear Transfer Coefficients

In the formulas below, $\kappa = k_m h$ is the dimensionless water depth, with k_m a mean wavenumber, where

$$(2\pi f_m)^2 = g k_m \tanh k_m h \quad (34)$$

The expressions for vertical displacement, Eulerian (fixed point) measurements are, in finite or infinite water depth (see [9] for more formulas):

$$T_{nb}^D(f_m) = c_{diff}(\kappa) k_m, \quad T_{nb}^S(\kappa) = c_{sum}(\kappa) k_m \quad (35)$$

$$\text{with } c_{diff}(\kappa) = \frac{\Pi(\kappa) + \kappa(1 - (\tanh \kappa)^2)}{\Pi(\kappa)^2 - 4\kappa \tanh \kappa} \quad c_{diff}(\infty) = 0 \quad (36)$$

$$\text{and } c_{sum}(\kappa) = \frac{1}{4} \left(\frac{2 + (1 - (\tanh \kappa)^2)}{(\tanh \kappa)^3} \right) \quad c_{sum}(\infty) = \frac{1}{2} \quad (37)$$

$$\text{where } \Pi(\kappa) = \tanh \kappa + \kappa(1 - (\tanh \kappa)^2) \quad \Pi(\infty) = 1 \quad (38)$$