
HAWAI project

WP3: Hydrodynamic wave loading

Optimisation of the time-series reconstruction

HAWAI

Ifremer report

WP3: Hydrodynamic wave loading

Optimisation of the time-series reconstruction

Marc Prevosto, Thomas Souldard

Contents

Optimisation of the time series reconstruction

Introduction,	1
Modeling of the low frequency behaviour,	3
Approximation of a quadratic system,	5
Synthesis of the low frequency transfer,	7
Approximation of the low frequency transfer,	10
<i>Example of QTF approximation,</i>	10
Steps of approximation,	12
Test case,	14
Assessment of the computing time,	15
<i>Number of operations,</i>	15
<i>Computing times,</i>	16
Assessment of the accuracy,	18
<i>Criteria of quality,</i>	18
<i>Effect of frequency sub-band,</i>	19
<i>Effect of the order of operations and of corrections,</i>	19
Note on the random frequency method,	26
Multi-directional QTF,	28
Conclusion,	30
References,	31
Appendix 1: Mean, variance and quantile comparisons,	32

Optimisation of the time series reconstruction

Introduction

If Newman's approximation is used to calculate the low frequency part of the non-linear transfer between the waves and the forces on an LNG carrier, the CPU time to make the time series reconstruction of wave loads is a small part of the total. However, using full QTF-matrices increases the required work by a factor equal to the number of wave components in the sea states and the total simulation time can be increased by 5 or 10.

Moreover, if we consider the multi-directional sea, the double summation becomes a quadruple summation at second order which is extremely time-consuming.

There is a way to accelerate this computation by making an algebraic or system approximation of the QTF. The aim of this report is to present the solutions, the problems and the performances of such system approximation.

After a brief presentation of the modeling of forces and response of a moored vessel, we will introduce a general approach of approximation of a quadratic transfer function.

A particular case of approximation consists of factorising the full matrix of QTF, considering only the main (highest modulus) eigenvalues and corresponding eigenvectors. We will see in using this approximation, that, the double summation with the number of waves multiplying the number of waves becomes that with the number of eigenvalues multiplying the number of waves.

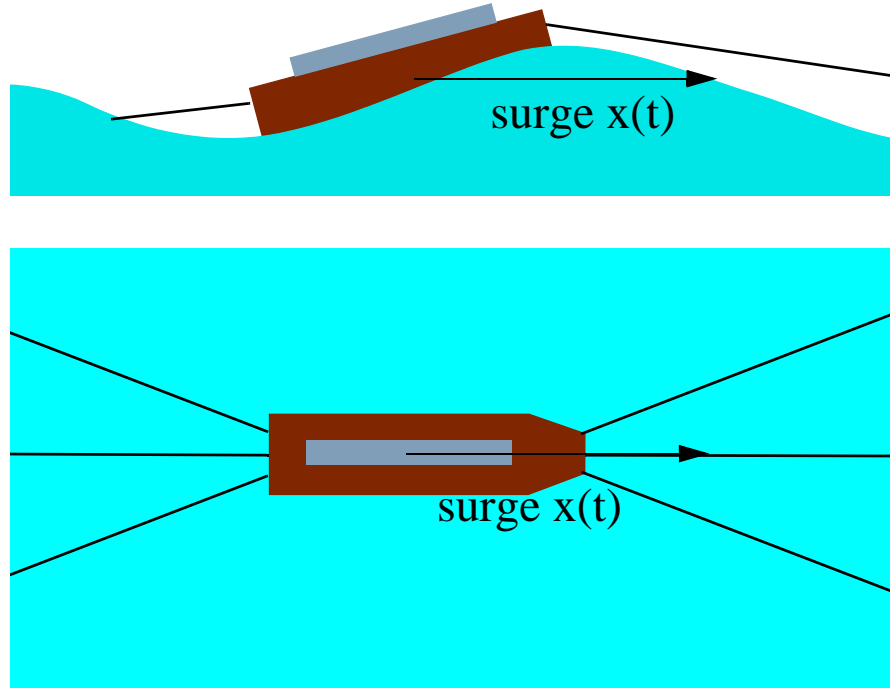
The gain is then equal to ratio between the number of waves and that of eigenvalues in uni-directional waves, while in multi-directional sea, the gain could be even more important (equal to the number of waves multiplying the number of headings and divided by the number of eigenvalues).

This study will compare the different approaches of this method on a case study furnished by Bureau Veritas. Extensive results are given for QTF uni-directional and some first results are commented for QTF multi-directional.

General considerations, uni- or multi-directional case, will be discussed on the different manners to run the approximation: eigendecomposition on the total QTF (including or not the wave spectrum), eigen-decomposition on a sub-frequency-band QTF, interpolation before or after approximation, corrections of the approximation of mean and standard deviation of the low frequency forces.

The assessment of the quality of the approximation will be presented on average and extreme distances. Some information on the number of operations and CPU time will be given.

Modeling of the low frequency behaviour



The behaviour of the vessel is modeled in a simplified way, first by a linear (RAO) plus a quadratic (QTF) transfer between the linear wave elevation $\eta(t)$ and the forces $c(t)$ (1) and secondly by a linear dynamic system between the forces and the movements $x(t)$ (2).

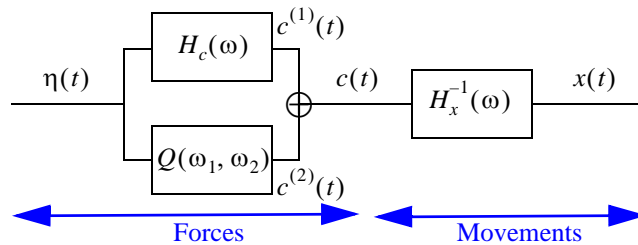
$$c(t) = (h_c(t) + q(t_1, t_2)) \otimes \eta(t) \quad (1)$$

$$\ddot{M}x(t) + D\dot{x}(t) + Kx(t) = c(t) = h_x(t) \otimes x(t) \quad (2)$$

The sign \otimes is for time convolution, $h_c(t)$, $h_x(t)$, the impulse response of the linear transfers and $q(t_1, t_2)$ the quadratic impulse response. M , D and K are respectively the mass, damping and stiffness coefficients.

In term of transfer function, the following diagram can summarize this modeling.

FIGURE 1 : Vessel behaviour transfer model



with $Q(\omega_1, \omega_2)$ the 2D Fourier transform of $q(t_1, t_2)$, and $H_c(\omega)$, $H_x(\omega)$ the Fourier transform of $h_c(t)$ and $h_x(t)$.

In fact, the problem that we study in this report is the optimisation of the time series reconstruction, neither $H_c(\omega)$ nor $H_x(\omega)$ make problem, but only the quadratic part and its low frequency components that is reduced to the difference terms when considering waves reduced to a high frequency band. In the same way, if the movement has to be simulated, $H_x(\omega)$ as a linear transfer must be left outside the optimisation.

The second order force time series can be written in function of the Fourier transform of the wave time series $\eta(\omega)$ as

$$c^{(2)}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(\omega_1, \omega_2) \eta(\omega_1) e^{j\omega_1 t} \eta(\omega_2) e^{j\omega_2 t} d\omega_1 d\omega_2 \quad (3)$$

which could be split into

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(\omega_1, \omega_2) \eta(\omega_1) e^{j\omega_1 t} \eta(\omega_2) e^{j\omega_2 t} d\omega_1 d\omega_2 = \\ & \int_0^{\infty} \int_0^{\infty} Q(\omega_1, \omega_2) \eta(\omega_1) e^{j\omega_1 t} \eta(\omega_2) e^{j\omega_2 t} d\omega_1 d\omega_2 + \\ & \int_{-\infty}^0 \int_0^0 Q(\omega_1, \omega_2) \eta(\omega_1) e^{j\omega_1 t} \eta(\omega_2) e^{j\omega_2 t} d\omega_1 d\omega_2 + \\ & \int_{-\infty}^0 \int_0^{\infty} Q(\omega_1, \omega_2) \eta(\omega_1) e^{j\omega_1 t} \eta(\omega_2) e^{j\omega_2 t} d\omega_1 d\omega_2 + \\ & \int_0^{\infty} \int_{-\infty}^0 Q(\omega_1, \omega_2) \eta(\omega_1) e^{j\omega_1 t} \eta(\omega_2) e^{j\omega_2 t} d\omega_1 d\omega_2 \end{aligned} \quad (4)$$

which simplifies in a sum part nonlinear interactions and difference part nonlinear interactions (low frequency part)

$$\begin{aligned} c(t) = c_+^{(2)}(t) + c_-^{(2)}(t) = 2\Re \left(\int_0^{\infty} \int_0^{\infty} Q(\omega_1, \omega_2) \eta(\omega_1) e^{j\omega_1 t} \eta(\omega_2) e^{j\omega_2 t} d\omega_1 d\omega_2 \right) + \\ 2 \left(\int_0^{\infty} \int_0^{\infty} Q(\omega_1, -\omega_2) \eta(\omega_1) e^{j\omega_1 t} \overline{\eta(\omega_2)} e^{-j\omega_2 t} d\omega_1 d\omega_2 \right) \end{aligned} \quad (5)$$

In the sequel we will be interested only in the difference part $c_-^{(2)}(t)$ of the forces and of the difference part of the QTF, $Q(\omega_1, -\omega_2)$.

$$c_-^{(2)}(t) = 2 \left(\int_0^{\infty} \int_0^{\infty} Q(\omega_1, -\omega_2) \eta(\omega_1) e^{j\omega_1 t} \overline{\eta(\omega_2)} e^{-j\omega_2 t} d\omega_1 d\omega_2 \right) \quad (6)$$

Approximation of a quadratic system

The simplest schemes of quadratic systems are of two types, squarer in time (Fig. 2) or diagonal-dominant in frequency (Fig. 3). Each of these schemes asks only for 1D calculations for simulations. For the squarer nonlinearity (we call it S in the sequel) the second order impulse response of the quadratic part is diagonal.

$$Q(\omega_1, \omega_2) \rightarrow q(t_1, t_2) = \delta(t_1 - t_2) \quad (7)$$

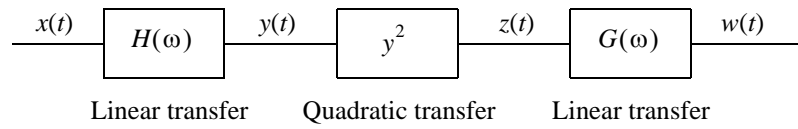
and corresponds to a QTF full of 1.

For the diagonal-dominant nonlinearity (we call it D in the sequel) it is the frequency transfer function itself which is (upper/lower) diagonal

$$Q(\omega_1, \omega_2) = \delta(\omega_1 - \omega_2 + k\Delta\omega) \quad (8)$$

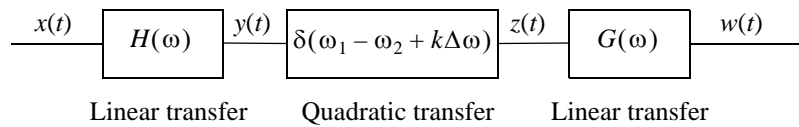
Rem. The system S is often called in system theory, L-N-L or Wiener-Hammerstein model (L-N-L for Linear dynamic-Nonlinear static-Linear dynamic), with here a static nonlinearity which is a squarer.

FIGURE 2 : Squarer nonlinearity system S



$$Q(\omega_1, \omega_2) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

FIGURE 3 : Diagonal-dominant nonlinearity D



$$Q(\omega_1, \omega_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A general quadratic transfer function (Fig. 4), as the one used for the modeling of low frequency forces or movements, if decomposed as a sum of a reduced number of such simple quadratic systems (Fig. 5), is shortly simulated. The simulation time is of course proportional to the number of systems necessary to well approximate the original quadratic transfer function.

A general quadratic system contains two linear transfer functions, one before the quadratic transfer and one after (Fig. 4). As these transfers do not increase really the complexity of the simulation, it is not always interesting to integrate these linear transfers in the global quadratic transfer function before the synthesis. In that case the global QTF is written (see [7])

$$Q_T(\omega_1, \omega_2) = H(\omega_1)H(\omega_2)Q(\omega_1, \omega_2)G(\omega_1 + \omega_2) \quad (9)$$

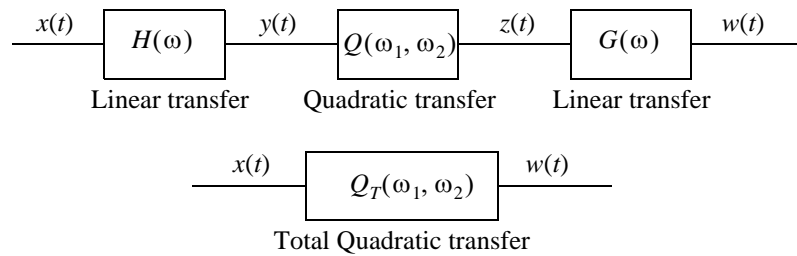
Sometimes, the fact to consider $Q_T(\omega_1, \omega_2)$ increases drastically the number of elementary systems necessary to well approximate the quadratic transfer. The choice of integration or not of H and/or G depends of the formulation of this linear transfers.

The general synthesis of a QTF with combination of S and D systems is very complicated and needs the resolution of a highly nonlinear problem. Meanwhile some progresses could be made in that sense in future works.

Two particular decompositions are simple to solve. A first classical one is used when the quadratic transfer function is diagonal dominant. In that case only the terms around the diagonal are kept and the system corresponds to the sum of a reduced number of D systems with

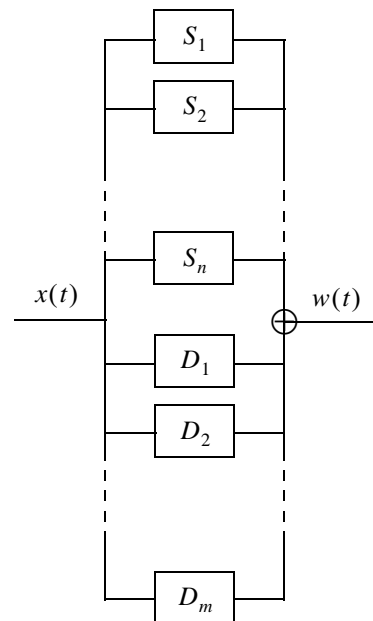
$$H_k(\omega) = Q(\omega_1, \omega_1 + k\Delta\omega), H_{-k}(\omega) = Q(\omega_1, \omega_1 - k\Delta\omega), G_k(\omega) = 1 \quad (10)$$

FIGURE 4 : General quadratic system



The second one is the combination in parallel of Wiener or L-N models (L-N for Linear dynamic-Nonlinear static). We will see hereafter how it is possible to synthesise the low frequency behaviour of the system as a sum of squarer systems (Fig. 2) with $G(\omega) = 1$

FIGURE 5 : S and D system decomposition



Synthesis of the low frequency transfer

The quadratic transfer function, which corresponds to the difference term, is Hermitian. If its eigendecomposition exists

$$\int_0^{\infty} Q(\omega_1, -\omega_2) \varphi_i(-\omega_2) d\omega_2 = \lambda_i \varphi_i(\omega_1) \quad (11)$$

the QTF can be written

$$Q(\omega_1, -\omega_2) = \sum_{i=1, \infty} \lambda_i \varphi_i(\omega_1) \overline{\varphi_i(\omega_2)} \quad (12)$$

with properties of orthogonality between the eigenfunctions $\varphi_i(\omega_1)$.

So the force $c_-^{(2)}$ in (6) can be expressed as a sum of simple quadratic systems

$$\begin{aligned} c_-^{(2)}(t) &= 2 \left(\int_0^{\infty} \int_0^{\infty} \sum_{i=1, \infty} \lambda_i \varphi_i(\omega_1) \overline{\varphi_i(\omega_2)} \eta(\omega_1) e^{j\omega_1 t} \overline{\eta(\omega_2)} e^{-j\omega_2 t} d\omega_1 d\omega_2 \right) \\ &= 2 \left(\sum_{i=1, \infty} \lambda_i \left(\int_0^{\infty} \varphi_i(\omega_1) \eta(\omega_1) e^{j\omega_1 t} d\omega_1 \right) \left(\int_0^{\infty} \overline{\varphi_i(\omega_2)} \eta(\omega_2) e^{-j\omega_2 t} d\omega_2 \right) \right) \\ &= 2 \left(\sum_{i=1, \infty} \lambda_i \left| \int_0^{\infty} \varphi_i(\omega_1) \eta(\omega_1) e^{j\omega_1 t} d\omega_1 \right|^2 \right) \end{aligned} \quad (13)$$

where the term into the modulus is the analytic signal associated to the output of a linear system $z_i(t)$

$$2 \int_0^{\infty} \varphi_i(\omega_1) \eta(\omega_1) e^{j\omega_1 t} d\omega_1 = z_i(t) + j\tilde{z}_i(t) \quad , \quad \tilde{z}_i(t) \text{ Hilbert transform of } z_i(t) \quad (14)$$

with

$$z_i(t) = \int_{-\infty}^{\infty} \varphi_i(\omega_1) \eta(\omega_1) e^{j\omega_1 t} d\omega_1 \quad (15)$$

Then $c_-^{(2)}(t)$ can be written

$$c_-^{(2)}(t) = \frac{1}{2} \sum_{i=1, \infty} \lambda_i (z_i(t)^2 + \tilde{z}_i(t)^2) \quad (16)$$

Practically, the frequency domain is discretised and considered only on a reduced frequency band, $\omega = -(N-1)\delta\omega, \dots, -\delta\omega, 0, \delta\omega, \dots, (N-1)\delta\omega$ and the time series is calculated as

$$c_-^{(2)}(t) = \sum_{k=-(N-1)\delta\omega}^{N-1} \sum_{l=-(N-1)\delta\omega}^{N-1} Q(k\delta\omega, l\delta\omega) \eta(k\delta\omega) e^{jk\delta\omega t} \overline{\eta(l\delta\omega)} e^{jl\delta\omega t} \delta\omega \delta\omega \quad (17)$$

When considering always the difference terms (low frequency terms)

$$c_-^{(2)}(t) = 2 \left(\iint_{0 \ 0}^{\infty \ \infty} Q(\omega_1, -\omega_2) \eta(\omega_1) e^{j\omega_1 t} \overline{\eta(\omega_2)} e^{-j\omega_2 t} d\omega_1 d\omega_2 \right) \quad (18)$$

$$c_-^{(2)}(t) = 2 \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} Q(k\delta\omega, -l\delta\omega) \eta(k\delta\omega) e^{jk\delta\omega t} \overline{\eta(l\delta\omega)} e^{-jl\delta\omega t} \delta\omega \delta\omega \quad (19)$$

which makes $c_-^{(2)}(t)$ periodic with a period $T = 2\pi/\delta\omega$ and sampled at the frequency $f_{\text{samp}} = 2N\delta\omega/2\pi$

It induces that (11) is a classical algebraic eigenproblem, which gives N eigenvalues λ_i and eigenvectors $\varphi_i(\omega)$.

$$c_-^{(2)}(t) = \frac{1}{2} \sum_{i=1, N} \lambda_i (z_i(t)^2 + \tilde{z}_i(t)^2) \quad (20)$$

and the $z_i(t)$, $\tilde{z}_i(t)$ are formulated as

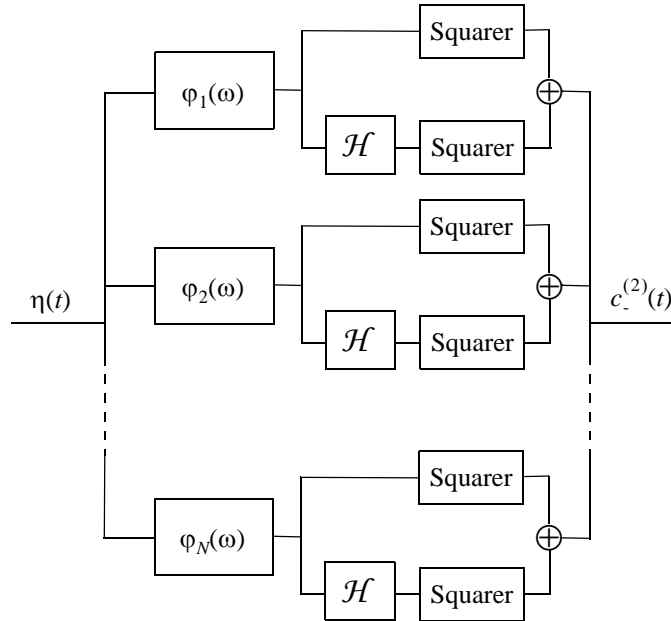
$$z_i(t) = \sum_{k=-(N-1)}^{N-1} \varphi_i(k\delta\omega) \eta(k\delta\omega) e^{jk\delta\omega t} \delta\omega \quad (21)$$

and

$$\tilde{z}_i(t) = \sum_{k=-(N-1)}^{N-1} (-j) \text{sign}(k) \varphi_i(k\delta\omega) \eta(k\delta\omega) e^{jk\delta\omega t} \delta\omega, \text{ with } j = \sqrt{-1} \quad (22)$$

So, (20) can be modelised as a combination of $2N$ L-N models (the Hilbert transform \mathcal{H} is a linear operator).

FIGURE 6 : Synthesis of the low frequency QTF transfer



The next step, that we will describe hereafter is to choose the dimension n , $1 \leq n \leq N$ of the eigen-subspace that will give a good approximation of the QTF,

that is to say, that gives an acceptable error $e(t)$ on $c_-^{(2)}(t)$. The definition of the norm of the error and what is acceptable left to be defined.

$$e(t) = c_-^{(2)}(t) - \frac{1}{2} \sum_{i=1, n} \lambda_i (z_i(t)^2 + \tilde{z}_i(t)^2) \quad (23)$$

All what have been described before can be generalized to directional waves and directional quadratic transfer function without theoretical problem.

$$\eta(t) = \int_{-\infty}^{\infty} \eta(\omega_1) e^{j\omega_1 t} d\omega_1 = \int_0^{2\pi} \int_{-\infty}^{\infty} \eta(\omega, \theta) e^{j\omega t} d\omega d\theta \quad (24)$$

and

$$c_-^{(2)}(t) = \frac{1}{2} \sum_{i=1, \infty} \lambda_i^\theta (z_i(t)^2 + \tilde{z}_i(t)^2) \quad (25)$$

with

$$z_i(t) = \int_0^{2\pi} \int_{-\infty}^{\infty} \varphi_i^\theta(\omega, \theta) \eta(\omega, \theta) e^{j\omega t} d\omega d\theta \quad (26)$$

Of course we could expect that the number of eigentransfers necessary to well approximate the QTF will be higher than in the non-directional case.

This technique of approximation of QTF has been now often used for the estimation of statistics of second order phenomena (second order waves, low frequency forces and response). See for example [1],[2],[3] or [5].

Approximation of the low frequency transfer

The Euclidian norm of a matrix is given by the highest singular value of the matrix. In the case of a Hermitian matrix, the singular values of the matrix are equal to the absolute value of the eigenvalues. So, if the eigendecomposition of the discretised QTF is

$$Q(\omega_1, -\omega_2) = \sum_{i=1, N+1} \lambda_i \varphi_i(\omega_1) \overline{\varphi_i(\omega_2)} \quad (27)$$

with

$$|\lambda_1| \geq |\lambda_2| \geq \dots |\lambda_n| \geq \dots \geq |\lambda_{N-1}| \geq |\lambda_N| \quad (28)$$

the best approximation is given by keeping the eigenvalues of highest modulus

$$\hat{Q}(\omega_1, -\omega_2) = \sum_{i=1, n} \lambda_i \varphi_i(\omega_1) \overline{\varphi_i(\omega_2)} \quad (29)$$

The norm of the matrix residue is given by $|\lambda_{n+1}|$

$$\begin{aligned} & \|Q(\omega_1, -\omega_2) - \hat{Q}(\omega_1, -\omega_2)\|_2 \\ &= \left\| \sum_{i=1, N+1} \lambda_i \varphi_i(\omega_1) \overline{\varphi_i(\omega_2)} - \sum_{i=1, n} \lambda_i \varphi_i(\omega_1) \overline{\varphi_i(\omega_2)} \right\|_2 \\ &= \left\| \sum_{i=n+1, N+1} \lambda_i \varphi_i(\omega_1) \overline{\varphi_i(\omega_2)} \right\|_2 = |\lambda_{n+1}| \end{aligned} \quad (30)$$

Example of QTF approximation

An illustration of the QTF approximation is given Figure 7.

The QTF was calculated on 200 frequencies with a frequency step $\Delta\omega = 0.001$ rad from the interpolation and restriction to the wave frequency band [0.35,0.55 rad/s] of an original QTF. The approximation obtained in keeping 3 eigenvalues from the 200 initial eigenvalues is very good (the figure shows the modulus of the QTF).

In Figure 8 we verify that the quality of the approximation is good. The absolute value of the eigenvalues decrease very rapidly and the norm of the QTF residue (the fourth eigenvalue) is very small compared to the norm of the QTF (the first eigenvalue).

FIGURE 7 : QTF (200 eigenvalues - left) vs approximated QTF (3 eigenvalues - right)

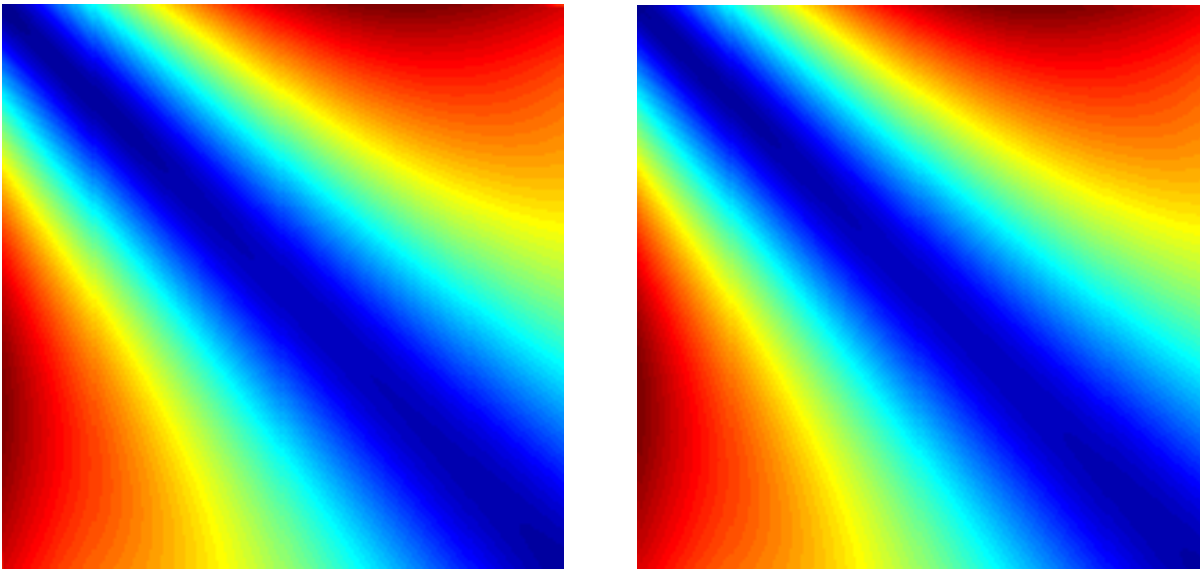
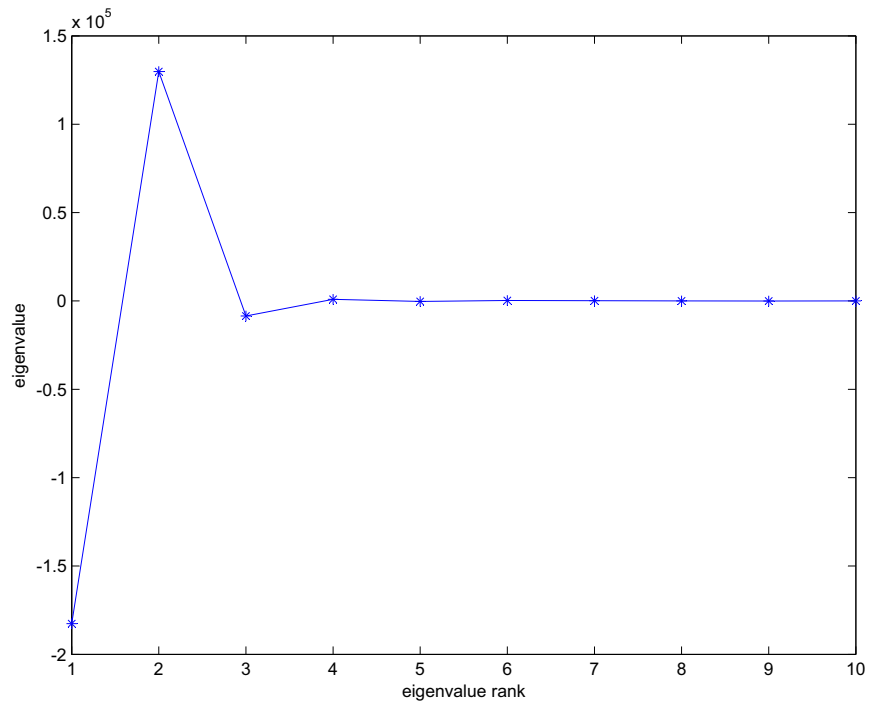


FIGURE 8 : QTF ten first eigenvalues



Steps of approximation

The step of approximation can be applied in different ways on the Quadratic Transfer Function.

Frequency sub-band. If the useful wave frequency band is a band defined by $[\omega_1, \omega_2]$ (no energy outside this band), it is more interesting to approximate (diagonalise) only the QTF restricted to this band

$$\left[\begin{array}{ccc} Q(0, 0) & Q(0, \Delta\omega) & \dots \\ Q(\Delta\omega, 0) & Q(\Delta\omega, \Delta\omega) & \dots \\ & & \dots \\ & & & \left[\begin{array}{ccc} Q(\omega_1, \omega_1) & & \\ & \dots & \\ & & Q(\omega_2, \omega_2) \end{array} \right] & \dots \end{array} \right] \quad (31)$$

This will decrease the number of eigenvalues necessary to reach the same level of approximation.

Interpolation. The QTF are calculated by the hydrodynamic software with a frequency step $\Delta\omega$. This step generally does not correspond to the step necessary for the simulation $\delta\omega$ as used in (17). In fact, generally $\delta\omega$ is smaller than $\Delta\omega$, and so it is necessary to interpolate to calculate the $Q(k\delta\omega, l\delta\omega)$. This interpolation can be made before or after the eigendecomposition. In that later case, the interpolation is made on the eigenvectors φ_i . We will choose in both the cases a linear interpolation.

So four methodologies are possible and will be compared in the sequel of this study.

- interpolation of raw QTF followed by eigendecomposition
- eigendecomposition of raw QTF followed by interpolation of eigenvectors
- extraction of sub-matrix QTF followed by interpolation of QTF, eigendecomposition
- extraction of sub-matrix QTF followed by eigendecomposition of sub-matrix QTF, interpolation of eigenvectors

Eigendecomposition. We used to find the eigenvalues and eigenvectors of the QTF an IRLM method (Implicit Restarted Lanczos method) [4]&[6]. This algorithm permits to calculate only the first eigenvalues of highest modulus of large matrix and their corresponding eigenvectors. It permits to gain computing time when the number of eigenvalues needed is very small compared to the size of the matrix, what is exactly the situation we have to deal with.

Correction. An improvement of the reconstruction of the time series can be obtained by the calculation of the theoretical mean and variance of the output of the quadratic transfer. It is very simple to see that the theoretical mean of (19) is given by

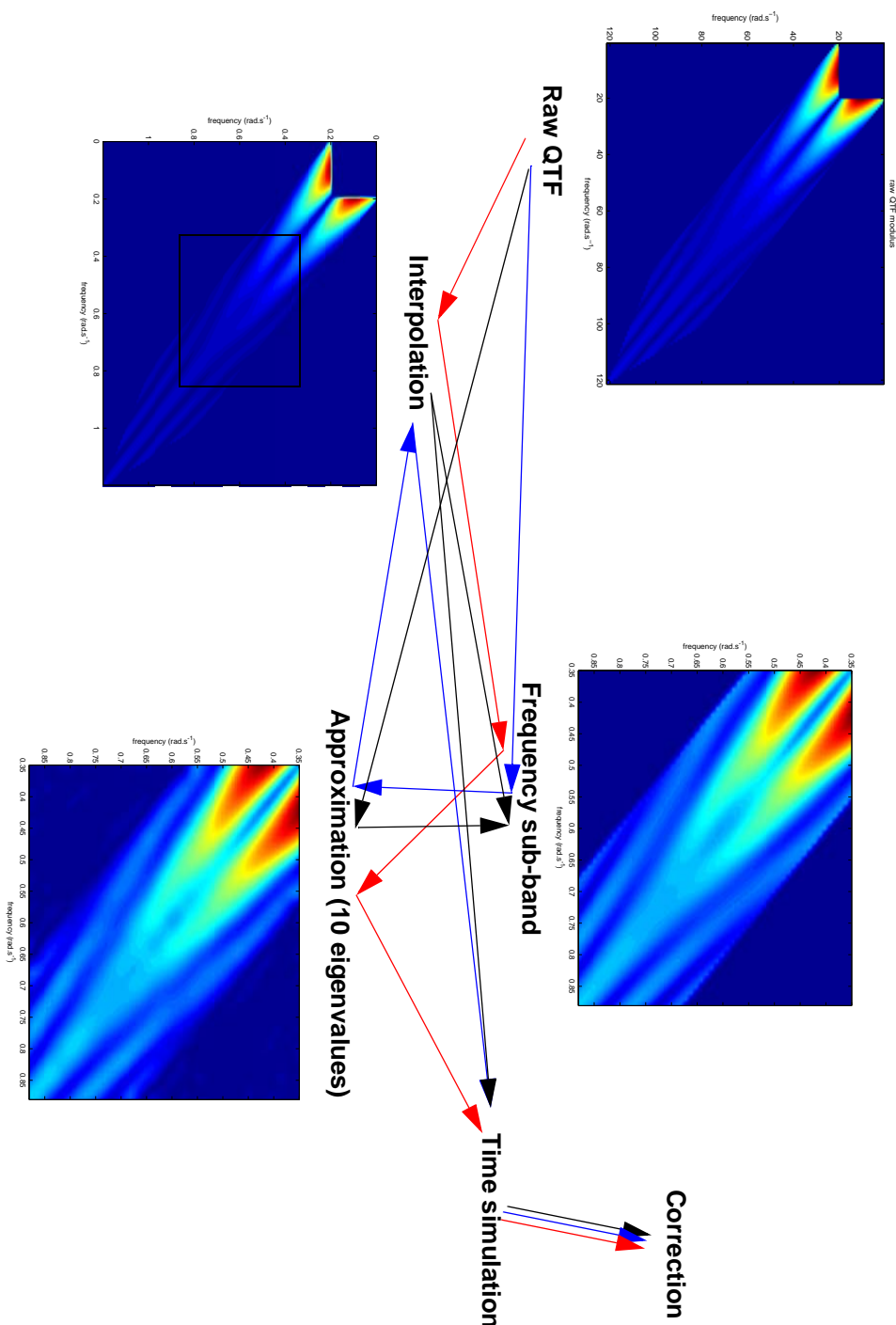
$$\langle c^{(2)}(t) \rangle = 2 \sum_{k=0}^{N-1} Q(k\delta\omega, -k\delta\omega) \eta(k\delta\omega) \overline{\eta(k\delta\omega)} \delta\omega \delta\omega \quad (32)$$

which is very simple to calculate. The approximation of Q , \hat{Q} (29) introduces an error on the mean that can be also calculated in the same way. So, a correction of the mean of the reconstructed time series can be applied.

In the same way, but with more complicated calculation, which are not given here, a correction of the variance can also be applied.

So, a lot of reconstruction schemes are possible and will produce different quality of approximation and computing times. Some are illustrated in Fig. 9 (red, blue and black paths).

FIGURE 9 : Reconstruction schemes



Test case

The test case that we used corresponds to the QTF of a vessel which is described in a Bureau Veritas HAWAI report. It corresponds to a vessel with a draft of 15m in 25m water depth. We only considered in this study the surge response.

The QTF are given from 0.2 rad.s^{-1} (0.032 Hz) to 1.2 rad.s^{-1} (0.191 Hz) with a step 0.01 rad.s^{-1} (0.0016 Hz). The modulus of the QTF that we used is given in Fig. 10.

The waves were described by a JONSWAP spectrum with parameters: $H_s = 5\text{m}$, $T_p = 12\text{s}$ and $\gamma = 10$ with a wave heading of 165° .

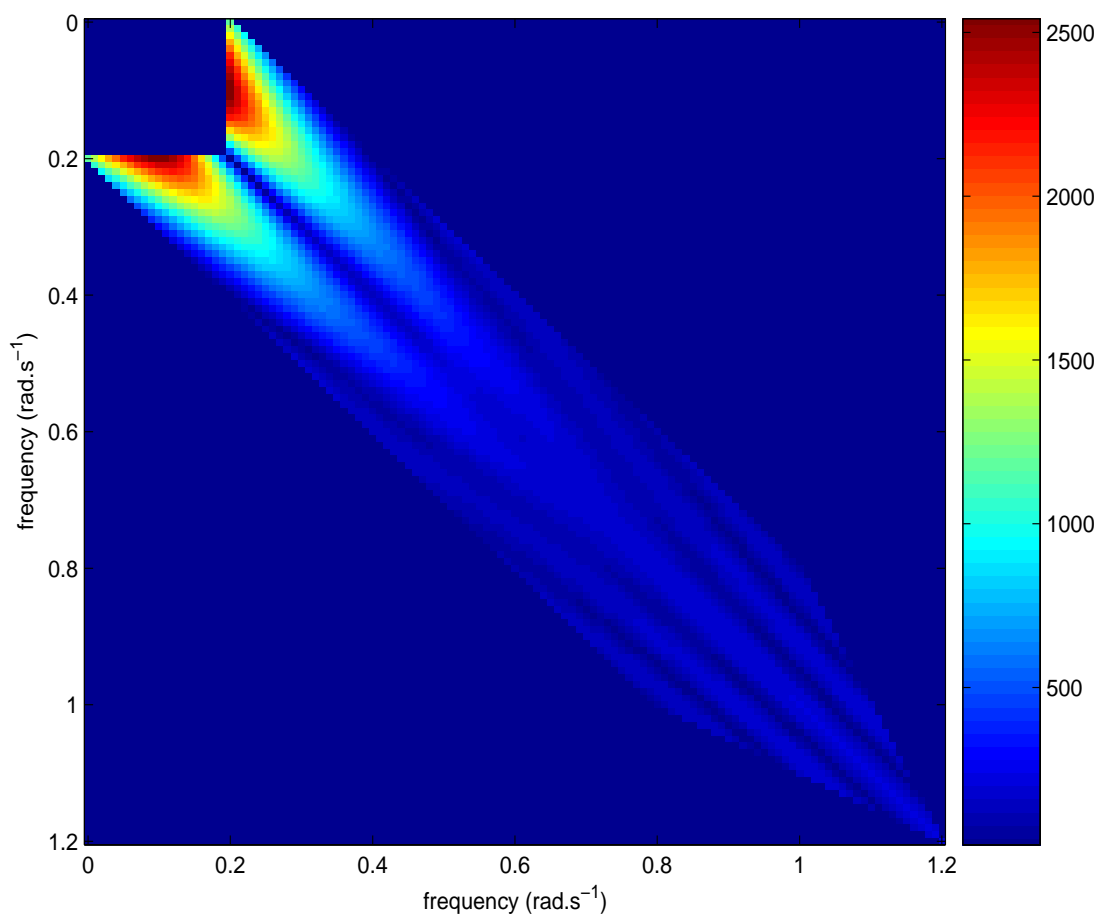
When used, the frequency sub-band is $[0.35, 0.88 \text{ rad/s}]$.

The dynamic transfer (2) has as coefficients:

$M = 257.9 \cdot 10^6 \text{ kg}$, $K = 1 \cdot 10^6 \text{ N.m}^{-1}$, $\xi = 0.08$, with ξ the critical damping, which gives a resonant pulsation $\omega = \sqrt{K/M} = 0.0622 \text{ rad.s}^{-1}$, then a resonant period $T_r = 100.9 \text{ s}$. The damping coefficient D is given by $2\xi\omega M$.

The time series are simulated with a sampling frequency 0.382 Hz on a duration which corresponds to 53617 seconds ($\sim 1\text{h}30$) when the number of points of the time series is 2048 .

FIGURE 10 : Modulus of the raw QTF



Assessment of the computing time

The counting of elementary operations is not always obvious and sometimes depends of the data itself, for example in convergent iterative schemes, as for example the eigendecomposition.

Anyway, we give hereafter some information about the relation between number of operations and size of vectors/matrices that we have to manipulate. In a second part we will give the computing times obtained on a PC.

Number of operations

First let us define:

- m_f the size of the raw QTF ($m_f \times m_f$)
- m_{fsb} the size of the QTF limited to the frequency sub-band
- k the ratio between the raw frequency step $\Delta\omega$ and the interpolated frequency step $\delta\omega$, $\delta\omega = k\Delta\omega$
- n the number of eigenvalues
- m_θ the number of directions (in the case of multi-directional QTF)
- npt the number of points of the time series simulation

Interpolation is common to all the methods and the number of operations depends at the same time of m_f (or m_{fsb}) and of k .

In the case of uni-directional QTF, $m_\theta=1$.

Approximated QTF

Eigenvalue decomposition with IRLM method takes $4.n.(k.m_\theta m_f)^2 + (30.n^2 + 18.n).k.m_\theta m_f + O(n^3)$ if a frequency sub-band is used $m_f=m_{fsb}$

This step of eigendecomposition is made one time for a given QTF. For each different sea-states or each different wave time series only the following step is necessary.

The *simulation* itself, (20), (21) and (22) with Fast Fourier Transform is proportional to $n.(m_\theta npt).log_2(m_\theta npt)$

Full simulation (Eq. (19)) with Fast Fourier Transform takes time proportional to $(m_\theta npt)^2.log_2(m_\theta npt)$

Dominant diagonal simulation

In the case of a dominant diagonal (19) can be restricted to some n_d diagonals around the main diagonal.

$$c_-^{(2)}(t) = 2 \sum_{d=-n_d}^{n_d} \sum_{k=0}^{N-1} Q(k\delta\omega, -(k+d)\delta\omega) \eta(k\delta\omega) \overline{\eta((k+d)\delta\omega)} e^{-jd\delta\omega t} \delta\omega \delta\omega \quad (33)$$

the computing time is then proportional to $(n_d.m_\theta).(m_\theta npt)$

For full and dominant diagonal simulations a step of preparation of calculation of the $e^{-jk\delta\omega t}$ is independent on sea-state and wave time series. This step (Table 2) is separated in the following tables from the time computing of simulation itself.

The interest of one or another method depends approximately, when npt is high, on the comparison between n , (n_d, m_θ) and (npt, m_θ) .

Computing times

We give hereafter (Tables 1-3) some computing times obtained with a PC (Intel Duo Core, 2.4GHz, 2Go RAM), and Matlab version 7.01.

Table 1 gives the CPU times necessary for the preparation of the QTF before the simulation itself. Table 2 gives the CPU times corresponding to the calculation of the $e^{-jk\delta\omega t}$ for the full and dominant diagonal methods. Table 3 gives the CPU times for 100 time series simulations of 1h30 with the three methods.

TABLE 1 : Approximated QTF, $k = 8.5$

	interpolat. (second)		eigendecomp. (second)		correction (second)	
	$k.m_f = 1024$	$k.m_{fsb} = 452$	$k.m_f = 1024$	$k.m_{fsb} = 452$	$k.m_f = 1024$	$k.m_{fsb} = 452$
$n = 40$	1.20	0.40	1.25	0.22	0.44	0.11
$n = 20$	1.20	0.40	1.03	0.16	0.36	0.08
$n = 10$	1.20	0.40	0.84	0.13	0.31	0.07
$n = 5$	1.20	0.40	0.48	0.09	0.31	0.06

TABLE 2 : Full and dominant diagonal preparation, $npt = 2048$

Full (s)			Dominant diagonal	
$k.m_f = 1024$	$k.m_{fsb} = 452$		$k.m_f = 1024$	$k.m_{fsb} = 452$
1.20	0.40	$n_d = 160$	1.25	0.22
		$n_d = 80$	1.03	0.16
		$n_d = 40$	0.84	0.13
		$n_d = 20$	0.48	0.09

TABLE 3 : Time series simulation - 100 time series, $npt = 2048$

full (s)		approximated QTF (s)		dominant diagonal (s)	
				$k.m_f = 1024$	$k.m_{fsb} = 452$
96	$n = 40$	1.7	$n_d = 160$	10.3	6.14
	$n = 20$	0.8	$n_d = 80$	5.3	3.12
	$n = 10$	0.4	$n_d = 40$	2.6	1.56
	$n = 5$	0.2	$n_d = 20$	1.3	0.8

The full simulation and approximated QTF simulation are calculated with Fast Fourier Transform and so are independent on a frequency sub-band.

It is clear that approximated QTF and dominant diagonal methods are very interesting in computing time compared to full 2D Fourier transform, as long as n and n_d

are small. We will see after that, in the uni-directional case, $n > 5$ gives very good results. But for the QTF we are working with, approximately $n_d = 160$ is necessary to have the same level of quality of approximation as $n = 10$ with the approximated QTF method (computing times in red in Table 3). This corresponds to a gain of 25 on the CPU time.

Assessment of the accuracy

Criteria of quality

To determine the quality of the reconstruction of the time series of forces and movements and the effect of the different choices of methodologies and parameters (interpolation before vs after, number of eigenvalues, ...) we have to choose one or more measures of the distance between the time series reconstructed and an "exact" time series.

We use two types of measure, a first one is average distance on the length of the time series (mean and standard deviation), the second one is a distance between statistics of extremes (maximum amplitude of the time series). They are defined hereafter

Average distances.

Relative mean distance

$$M = \mathbb{E}(x_r(t) - x_R(t)) / \mathbb{E}(x_R(t)) \quad (34)$$

where \mathbb{E} stands for statistical expectation, $x_R(t)$ (resp. $x_r(t)$) is the Reference (resp. reconstructed) time series. In fact it corresponds to the relative error on the drift force or offset displacement.

Relative standard deviation distance

$$E = \sqrt{\mathbb{E}(\overline{(x_r(t) - x_R(t))^2}) / \mathbb{E}(\overline{x_R(t)^2})} \quad (35)$$

with \bar{x} the centered time series

$$\bar{x} = x - \mathbb{E}(x)$$

Extreme distances.

High level quantiles of the maximum value of a time series (1h30) are calculated for the reference time series and for the approximated one.

A quantile q_x is defined from the cumulative distribution of the maximum as

$$P(X_{max} < q_x) = x \quad (36)$$

Three levels of quantiles have been analysed for x equals 0.999, 0.99 and 0.9.

When we are looking at negative extreme values the definition is changed as

$$P(X_{min} > q_x) = x \quad (37)$$

It is this definition that in fact we will use in our results.

This criteria of goodness of the high quantiles permits to assess the ability of the approximation to well reproduce extreme behaviour of the system.

All the figures have been calculated after the simulation of 40000 time series for each cases. This high number of simulations is particularly necessary for a good estimation of extreme statistics. The case 50 eigenvalues is taken as the reference as there is not any difference on the previous distances between 50 and 70 eigenvalues. The complete values of these different criteria in the different cases of approximation are given in Appendix 1. An example of comparison of the time series with different choices of number of eigenvalues is given in Figure 12. The quality of the case 50 eigenvalues is also illustrated by the Figure 13, where it is compared to the full 2D Fourier computation (19). Visually the difference is not

perceptible, and if the time series of the difference of the forces is plotted (Figure 13, bottom), the residue is close to the order 10^1 N, compared to the order of 10^6 N for the forces, which is quite excellent.

Effect of frequency sub-band

If the choice of a frequency sub-band limits the application of the approximation to cases where the energy of the waves are in this sub-band, it permits at equal quality to decrease the number of eigenvalues and so to decrease the time of simulation.

If, for example, we look at the relative residue of the approximation on QTF (see Eq. (30)), which corresponds to the ratio of modulus of the first neglected eigenvalue to the largest one

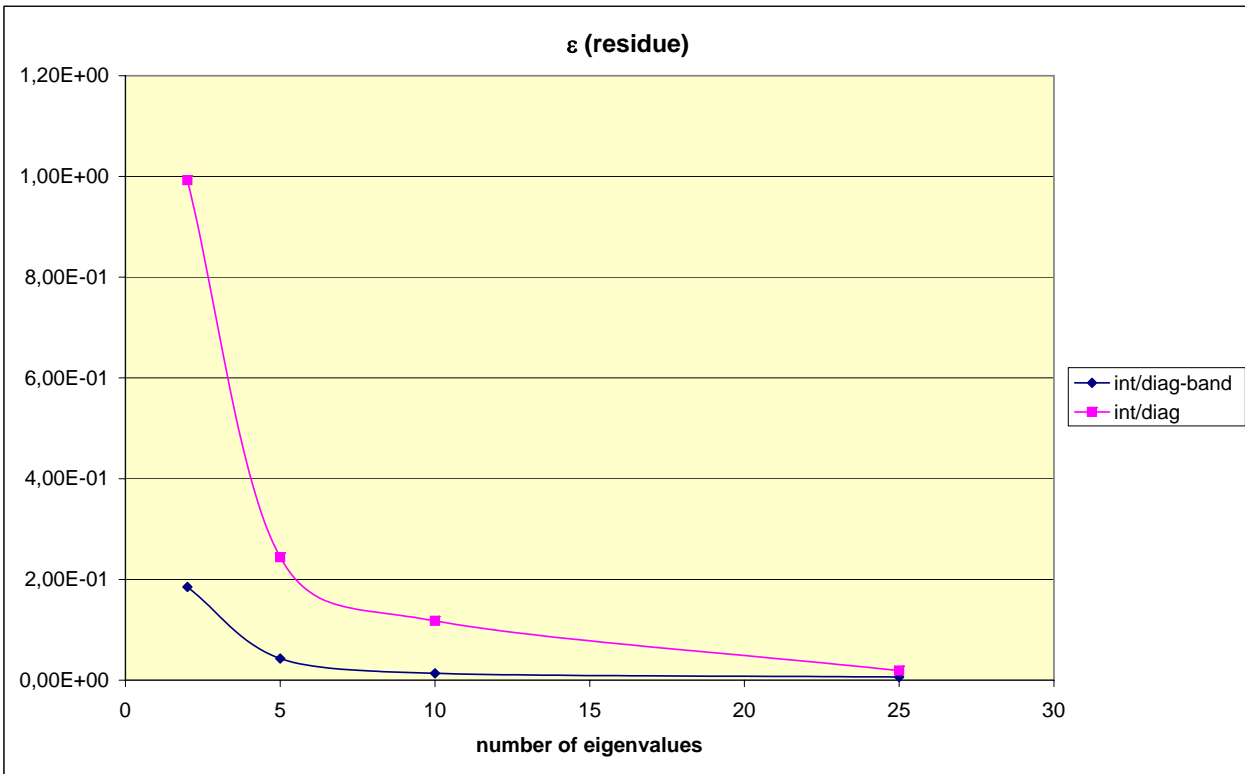
$$\varepsilon = \frac{\|Q_T - \hat{Q}_T\|_2}{\|Q_T\|_2} \quad (38)$$

where Q_T and so \hat{Q}_T integrate the wave spectral density

$$Q_T(\omega_1, -\omega_2) = Q(\omega_1, -\omega_2)S(\omega_1)S(\omega_2) \quad (39)$$

we see in Fig. 11, that in the case of interpolation followed by diagonalisation the gain in the quality is very important for reduced number of eigenvalues. It could be verified in Appendix 1, that this ε is a good indicator of the quality of the approximation.

FIGURE 11 : Effect of the frequency sub-band on QTF norm residue



Effect of the order of operations and of corrections

In Figs. 14-17 are plotted the relative mean and standard deviation (Eq. 34 & 35) as function of the number of eigenvalues in the case of frequency sub-band. Figs. 14-15 for the forces and Figs. 16-17 for the displacements. These figures show, on the mean as on the standard deviation, a clear degradation of the approximation

between interpolation followed by diagonalisation and diagonalisation followed by interpolation. If diagonalisation followed by interpolation permits a gain in CPU time (the eigendecomposition is made on a smaller matrix) it is paid by a loss of quality or CPU time in the simulation.

Meanwhile in the case of multi-directional seas and QTF, it permits to work with smaller matrices, to decrease CPU time of eigendecomposition and to avoid problems of memory space.

The effect of corrections on mean and standard deviation of the time series is also clear. For the mean the correction is total as the mean of the difference is the difference of the mean, so the mean error due to approximation is completely corrected. But for the standard deviation of the error, the effect of correction is clearly visible only for very low number of eigenvalues (<5). Anyway, for 5 eigenvalues the standard deviation of the error of approximation is only 5% for the forces and 3% for the displacements.

We could note that there is no difference between the relative mean error on forces and displacements. This result is obvious as the mean of the displacement is obtained by a simple multiplication of the mean of forces with $H_x^{-1}(0)$ (see Eq. 2 & Fig. 1).

An interesting result concerns the choice of frequency sub-band. We can note in the tables of Appendix 1 that, when we use the entire frequency band, the results are less accurate. For example, taking 25 eigenvalues on the entire frequency band is equivalent to 10 eigenvalues with the frequency sub-band, although, after the approximation of the QTF, the 2nd order transfer function will be multiplied by the limited frequency band of the waves. So, it is important, if possible, to well define the useful frequency band of the QTF to optimise the reconstructions.

Quantile. The results on quantile 0.999 given in Fig. 18&19, show again, that also on extremes values, the effect of approximation becomes very weak when the number of eigenvalues is higher than 5, with a tendency for the displacements (Fig. 19) to underestimate the extreme values.

Spectral density. Time series simulations of good quality ask also to respect the distribution of the variance in frequency. Actually, the forces are the input of resonant systems, and the behaviour of these systems are mainly driven by the distribution of variance in the frequency band around the resonant frequency. Fig. 20 gives the spectral density of the force for four different approximations. If clearly the frequency distribution is very bad for two eigenvalues, 5 eigenvalues is not so bad and 10 eigenvalues is quite perfect. When the correction of the variance is applied (Fig. 21), the 5 eigenvalues case is even better.

FIGURE 12 : Simulation of forces with 50, 10 and 2 eigenvalues

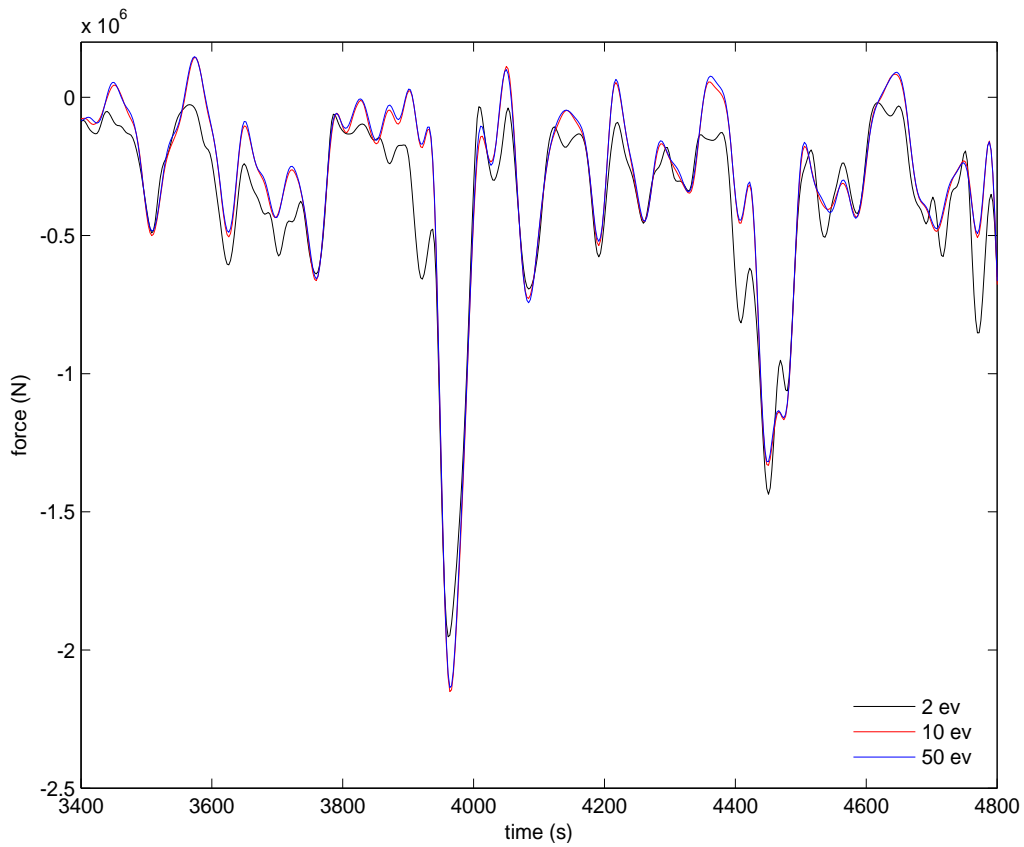


FIGURE 13 : Simulation of forces, 50 eigenvalues vs. full computation

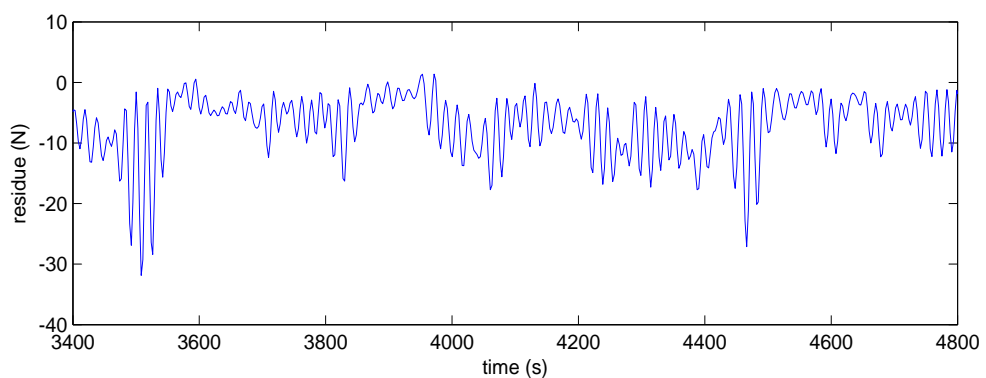
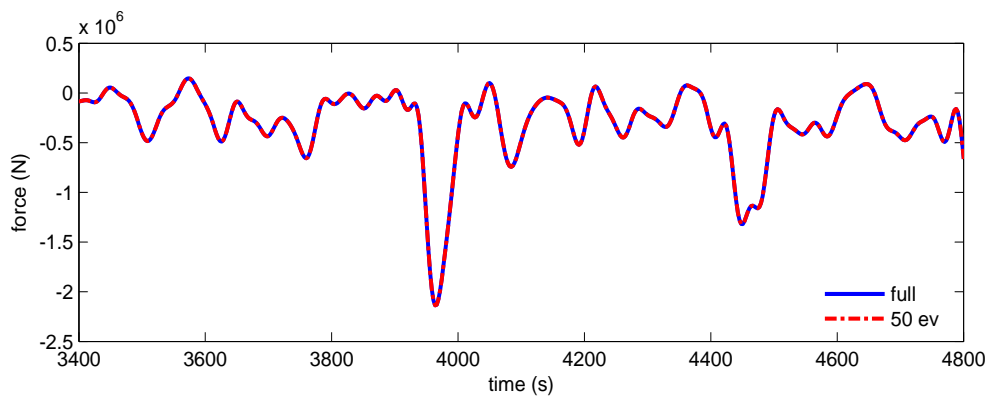


FIGURE 14 : Comparison of different methods - Forces - Relative Mean

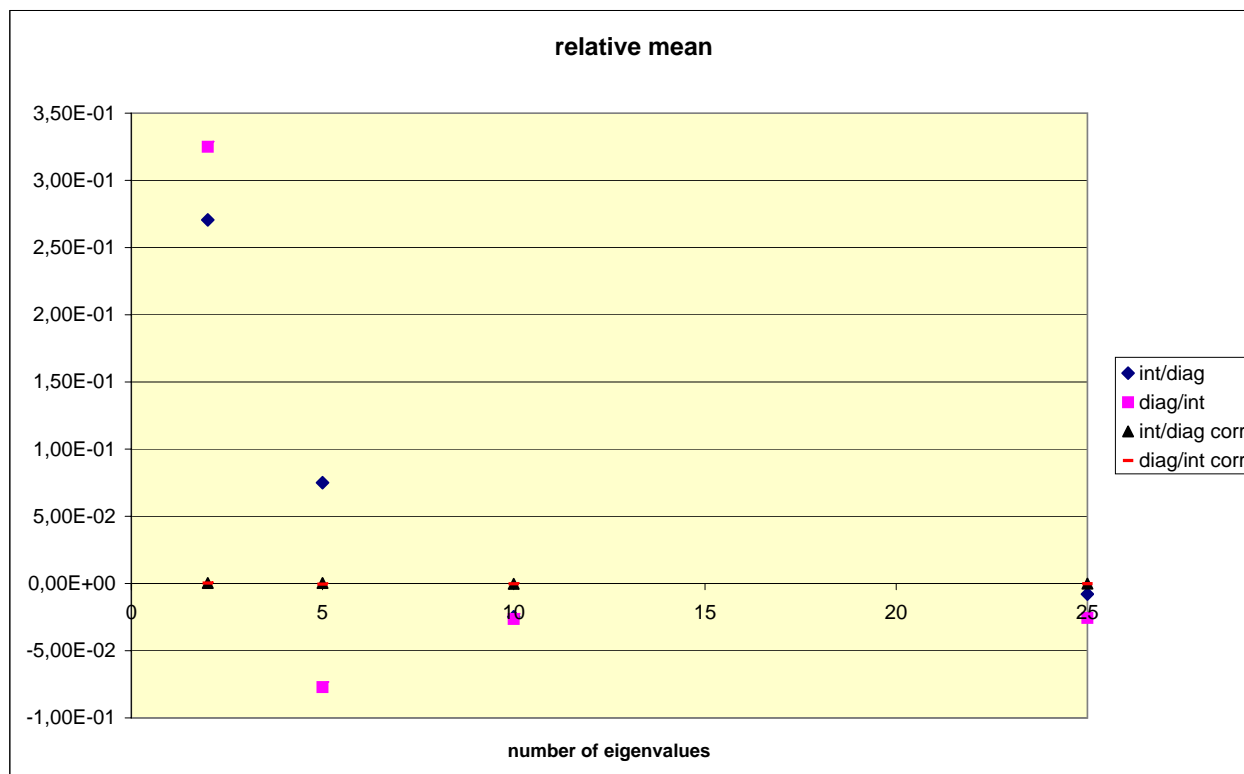


FIGURE 15 : Comparison of different methods - Forces - Relative Standard deviation

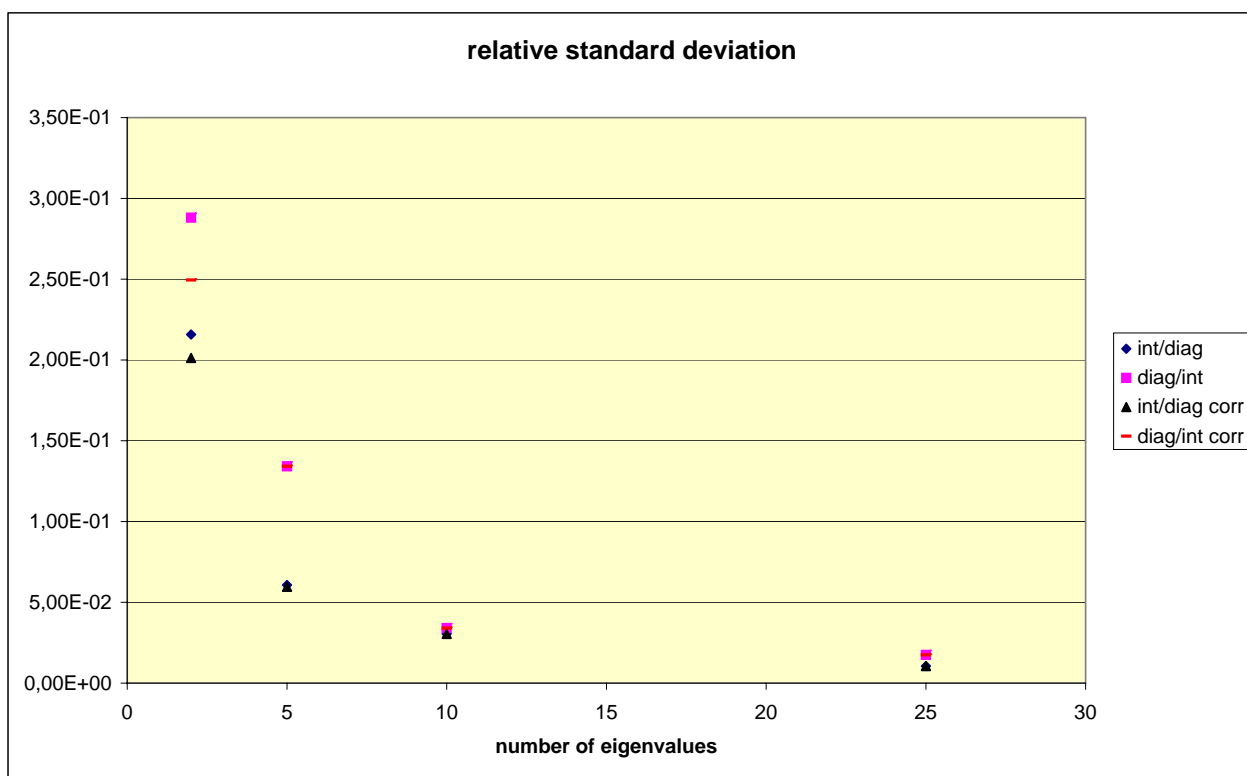


FIGURE 16 : Comparison of different methods - Displacements - Relative Mean

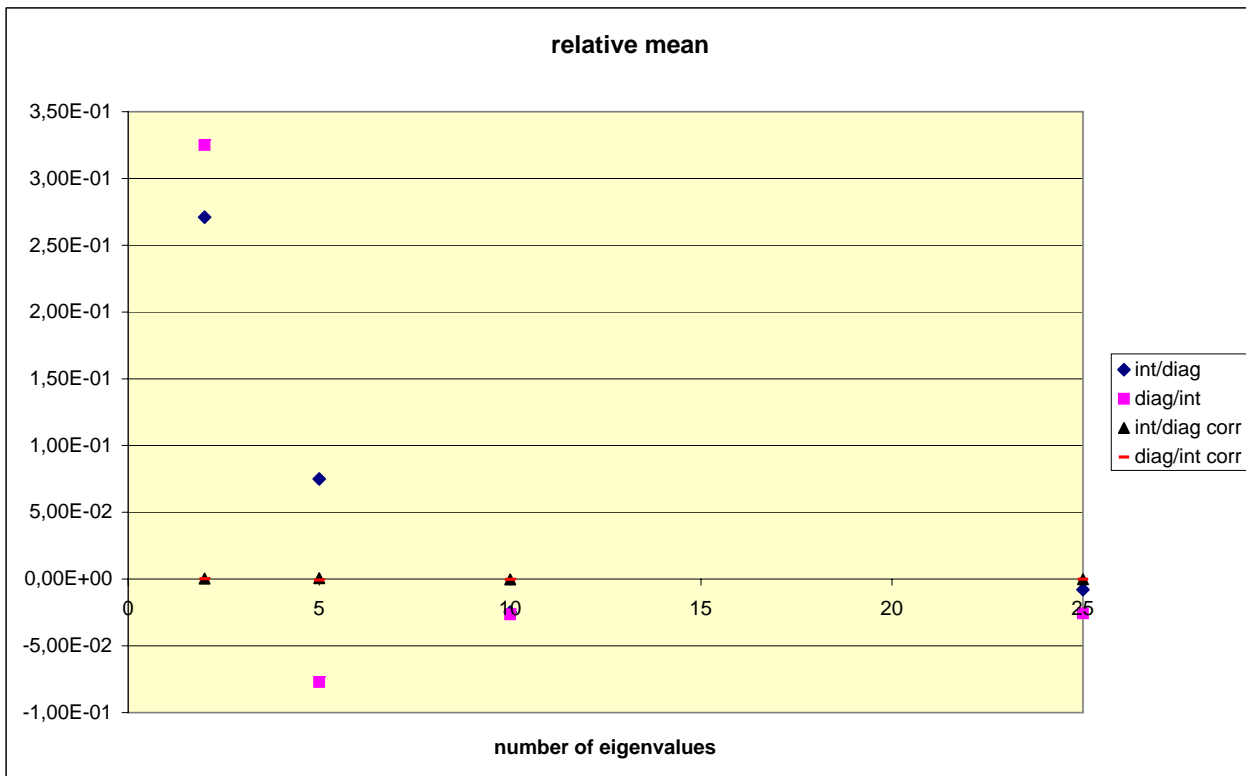


FIGURE 17 : Comparison of different methods - Displacements - Relative Standard deviation

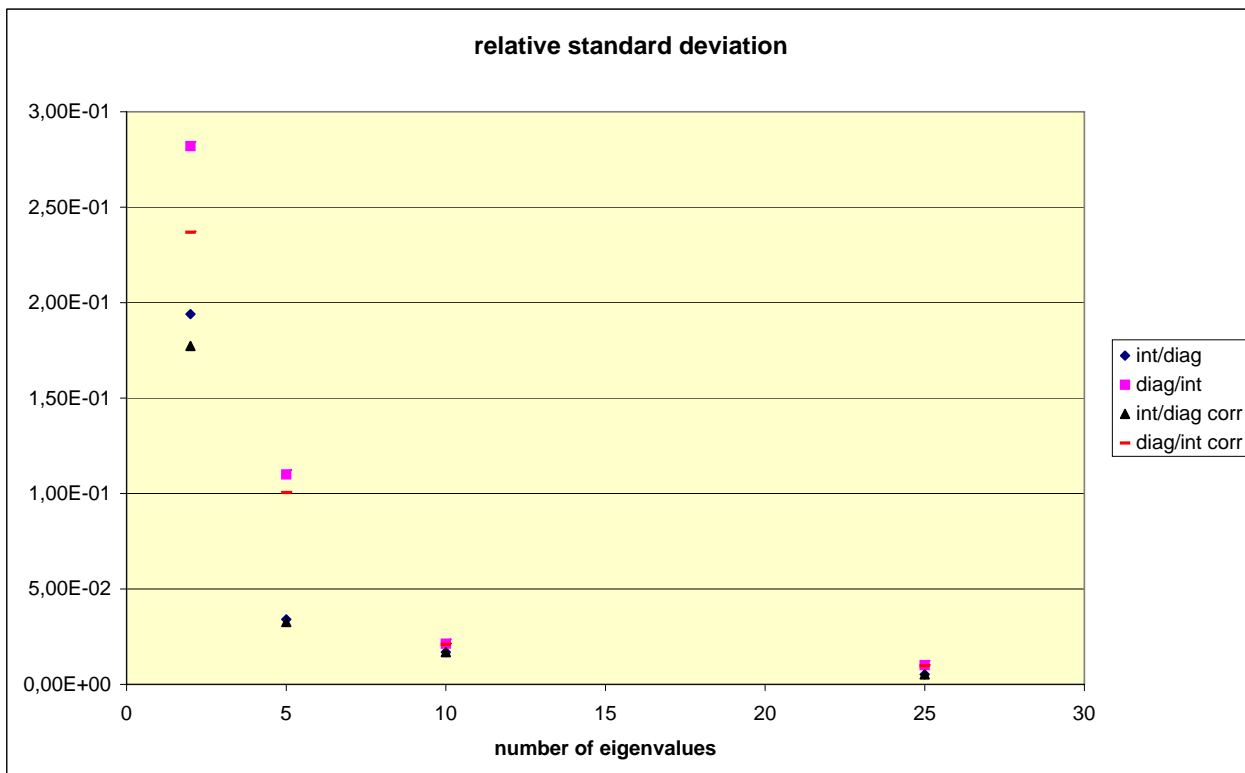


FIGURE 18 : Comparison of different methods - Forces - Quantile

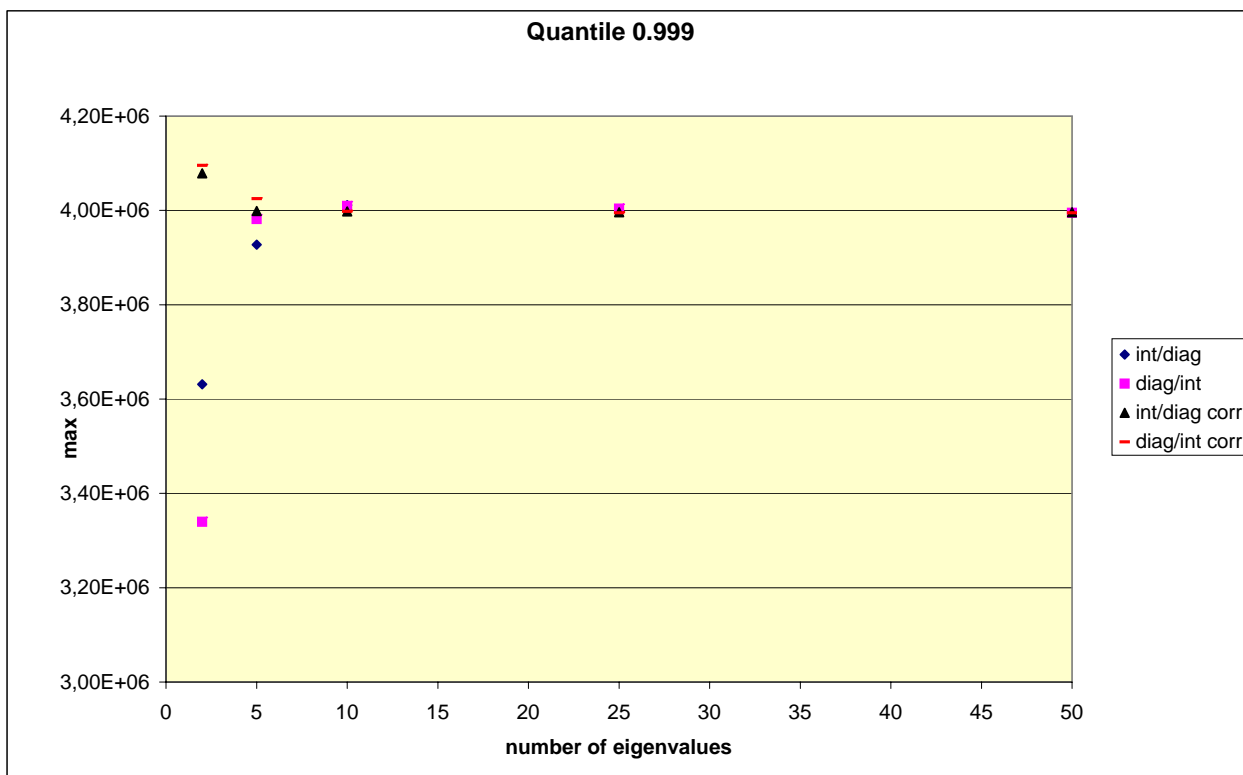


FIGURE 19 : Comparison of different methods - Displacements - Quantile

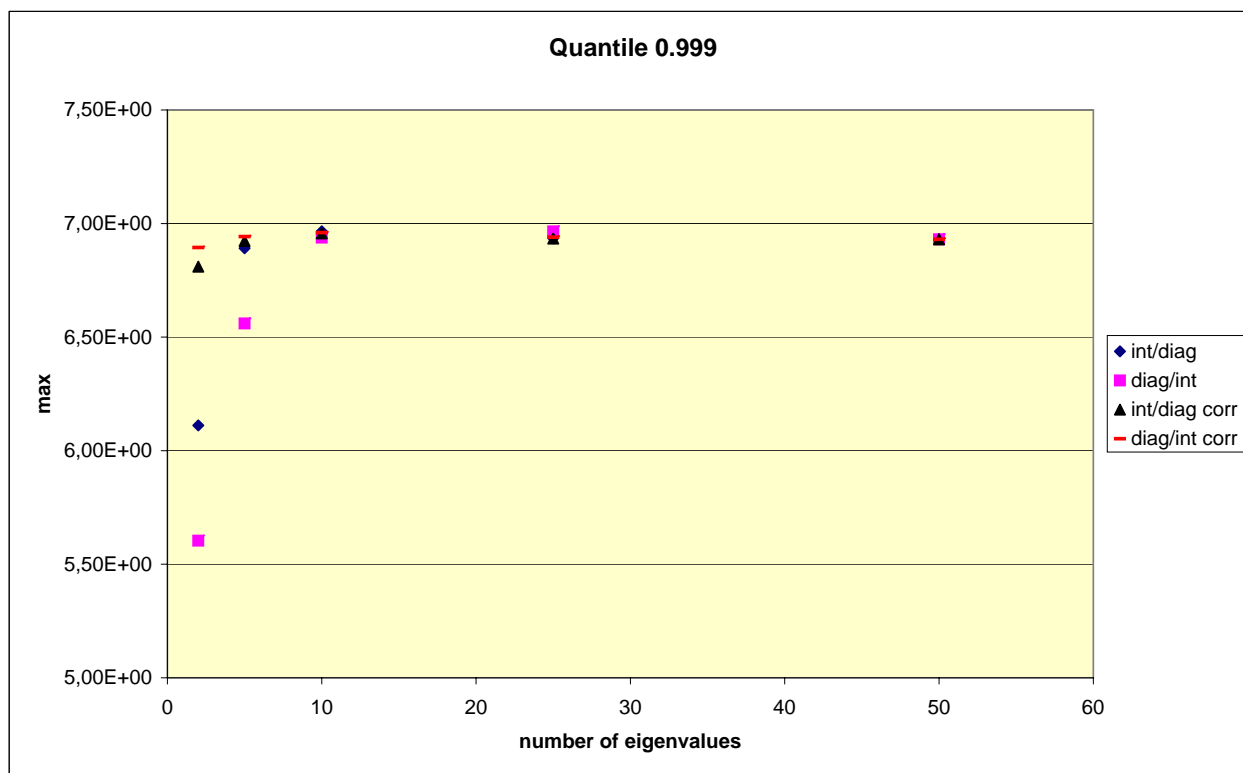


FIGURE 20 : Spectral density of forces - 50, 10, 5 and 2 eigenvalues

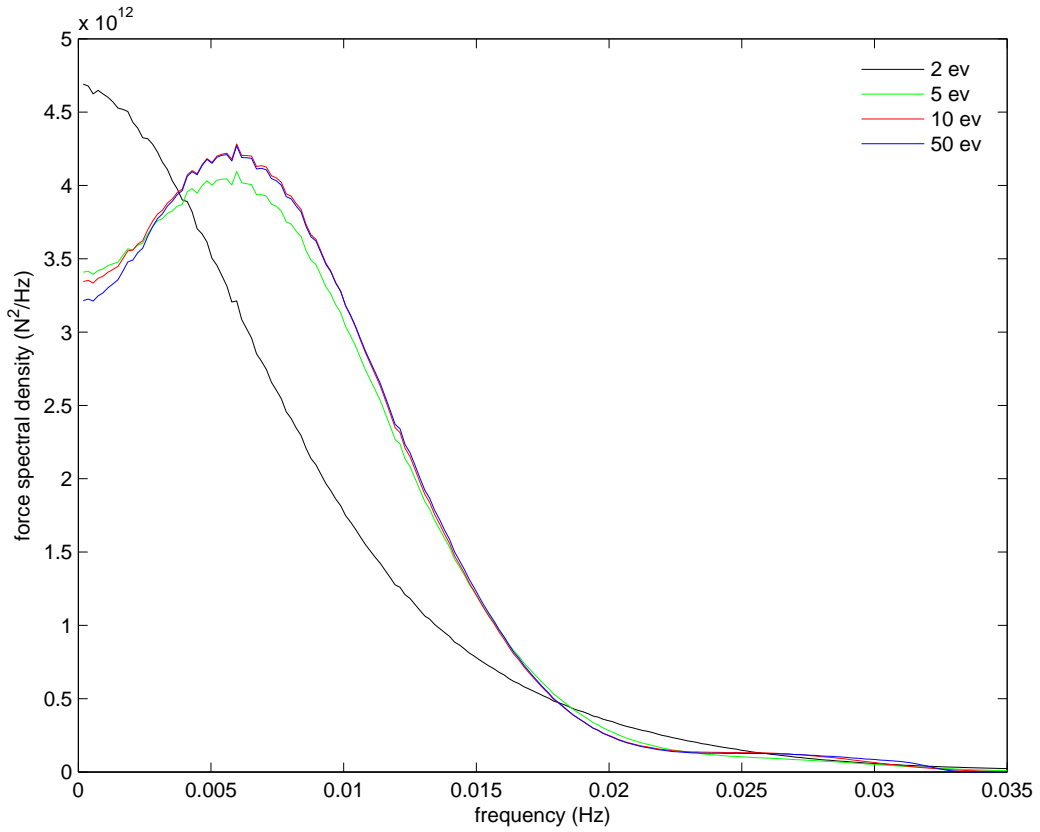
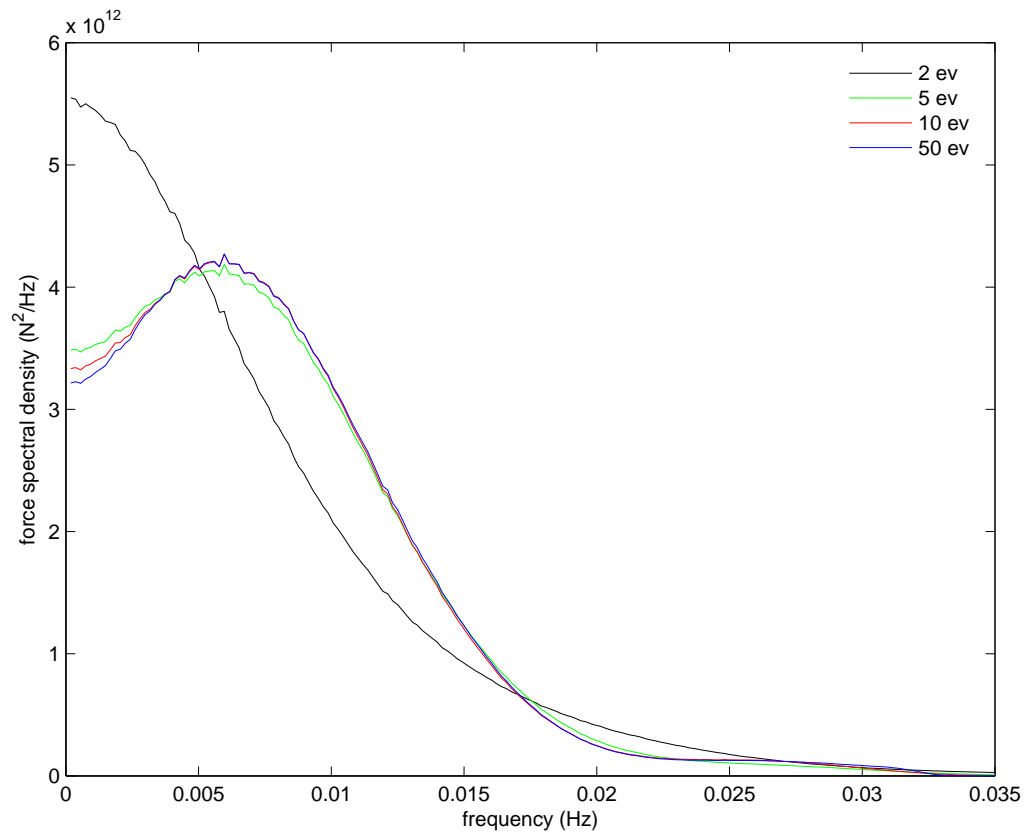


FIGURE 21 : Spectral density of corrected forces - 50, 10, 5 and 2 eigenvalues



Note on the random frequency method

A now classical method to simulate long time series with a limited number of operations is to use random frequency simulation.

$$c_{-}^{(2)}(t) = 2 \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} Q(k\delta\omega, -l\delta\omega) \eta(k\delta\omega) e^{jk\delta\omega t} \overline{\eta(l\delta\omega)} e^{-jl\delta\omega t} \delta\omega \delta\omega \quad (40)$$

In fact, in a simplified way, in (40), the frequencies $k\delta\omega$ in $e^{jk\delta\omega t}$ are not taken regularly spaced, but as $(k-1)\delta\omega + \Omega$, where Ω is a random variable uniformly distributed in $[-\delta\omega/2, \delta\omega/2]$ which is drawn independently on each frequency. This will permit to reduce the number of components N , in increasing $\delta\omega$, keeping the same duration of the time series without periodicity of this time series. Then, It permits to reduce drastically the computing time without reducing the length of the time series.

An example is given Figure 22 with the simulation of a wave time series. It corresponds to the first 2400 seconds of 3 hour simulation with a sampling frequency f_{samp} equals 1 Hz. The "purely random" corresponds to

$$\eta(t) = \sum_{k=-(N-1)}^{N-1} \eta(k\delta\omega) e^{jk\delta\omega t} \delta\omega \quad (41)$$

with $npt = 2N = 3.3600.f_{samp}$, $\delta\omega = 2\pi f_{samp}/npt$

The "periodic" corresponds to the same formula but with $\delta\omega = 2\pi f_{samp}/200$, which makes $\eta(t)$ periodic with a period $T = 200/f_{samp}$. The "random frequency" uses the same $\delta\omega$ but with random frequencies as explained before.

The "random frequency" method, as expected, produces non-periodic time series, but when we calculate the correlation function (Figure 23) we observe that, for high time lags, the waves are not uncorrelated as it must be. That is to say that, when calculating statistics related to the duration of a sea-state (here 3 hours), for example the maximum crest, these statistics will be biased, and in such a way that they will be much closer to statistics corresponding to the duration T . The conclusion would be the same on $c_{-}^{(2)}(t)$.

This effect, of course decreases as T is increased. Anyway, if necessary, this method could be coupled with the approximation method (Eq. ,21&22) if it is necessary to decrease more the computation time of time series.

FIGURE 22 : Time series - periodic (top), random frequencies (middle), purely random (bottom)

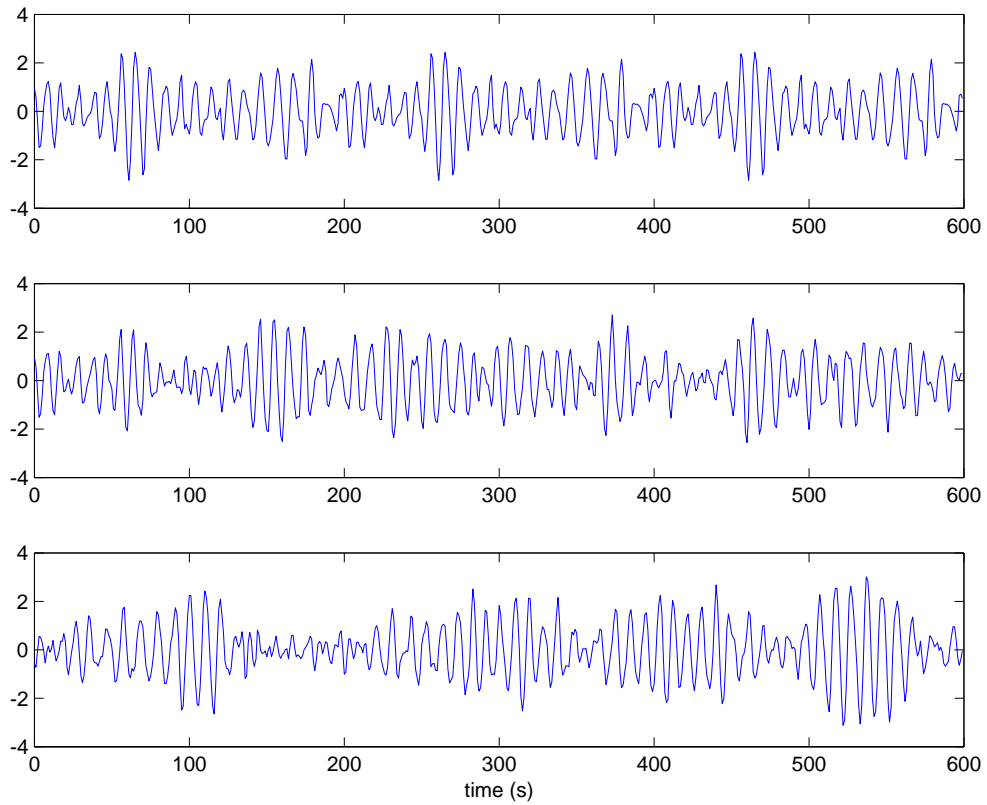
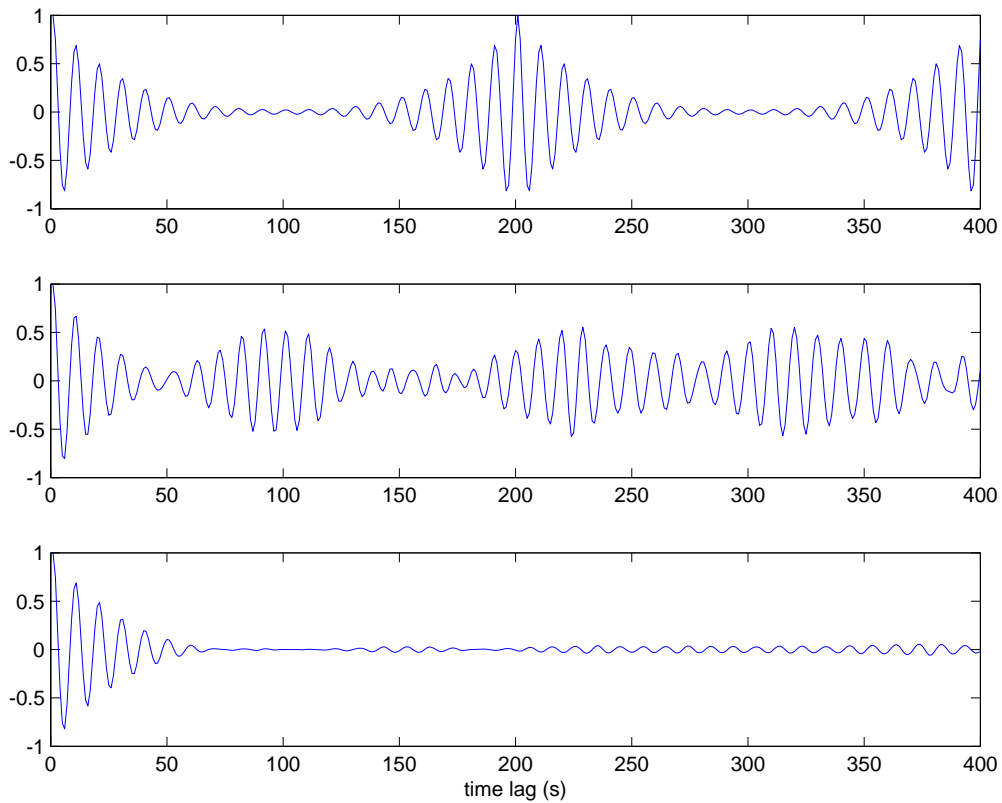


FIGURE 23 : Correlation functions - periodic (top), random frequencies (middle), purely random (bottom)

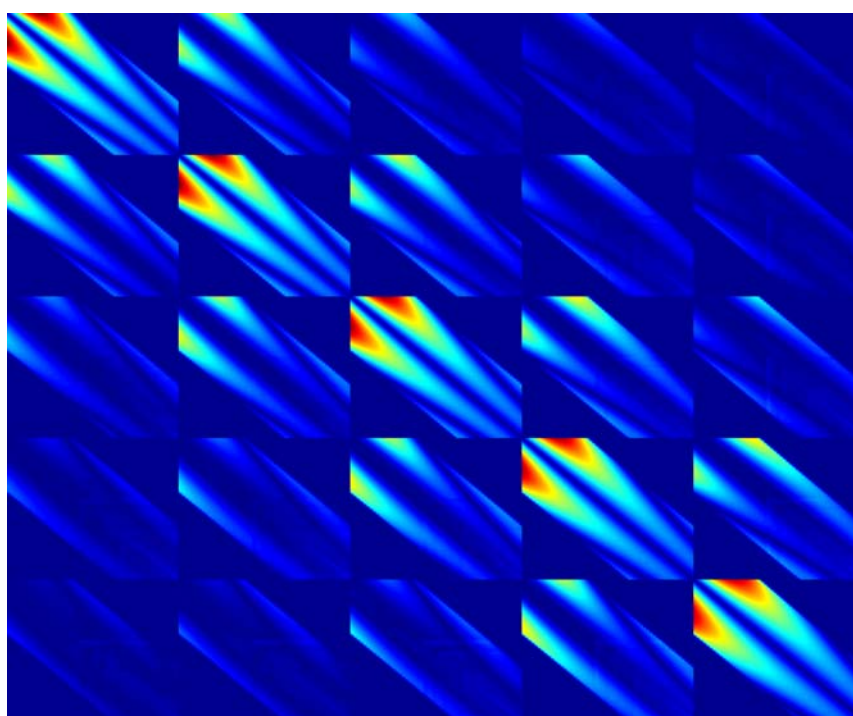


Multi-directional QTF

As it was seen before (Eqs (24)-(26)), the extension of the method of approximation of the QTF in the case of directional QTF and waves is quite direct. The dimension of the QTF is multiplied by the number of directions which are considered. An example of QTF for 5 heading directions is given in Fig. 24.

One problem that could be encountered is that the frequency interpolated directional QTF has a size not compatible with the memory of the computer, mainly for the eigendecomposition. One solution is to choose to interpolate after the eigendecomposition. As we commented before this method induces a light loss of accuracy at equal number of eigenvalues, but it is sometimes the only solution and it also permits to reduce drastically the CPU time for the eigendecomposition.

FIGURE 24 : Modulus of the raw directional QTF on the frequency sub-band



To analyse the accuracy of approximation obtained in the directional case, we have compared the relative Euclidian norm of the residue of approximation of the QTF

$$\varepsilon = \|\text{QTF}_{x_{ev}} - \text{QTF}\|_2 / \|\text{QTF}\|_2 \quad (42)$$

which, as we discussed before, is a good indicator of the quality of the reconstructed time series. $\text{QTF}_{x_{ev}}$ indicates QTF approximated with x eigenvalues.

Two different directional QTFs have been considered, one with 5 heading directions $[0^\circ, 10^\circ, 20^\circ, 30^\circ, 40^\circ]$ (Fig. 24) and a second one with 10 heading directions $[0^\circ, 10^\circ, 20^\circ, 30^\circ, 40^\circ, 50^\circ, 60^\circ, 70^\circ, 80^\circ, 90^\circ]$. The method with frequency sub-band and interpolation before eigendecomposition has been used. The results have been compared to the uni-directional case (heading=0°).

Figure 25 shows the relative residue for these three cases in function of the number of eigenvalues of the approximation. These curves indicates that the number of eigenvalues necessary to keep the same accuracy as in the uni-directional case corresponds to multiply the number of eigenvalues of the uni-directional case by a number slightly lower than the number of heading directions.

In a different way, we have plotted in Figure 26, the ratio, at a given residue, between the number of eigenvalues in the directional case (5 and 10 dir.) and this number for the unidirectional case. It shows, in a strange way, that the ratio reaches a maximum around $\varepsilon=3.10^{-2}$ and decreases for higher accuracies to reach approximately the number of heading directions multiply by 3/5.

In using the figures of part "Assessment of the computing time" and "Assessment of the accuracy", we can estimate that, in the case of 10 heading directions, 100 seconds will be necessary to simulate 100 time series of 1h30, with an accuracy of 10^{-2} .

FIGURE 25 : Relative residue vs number of eigenvalues

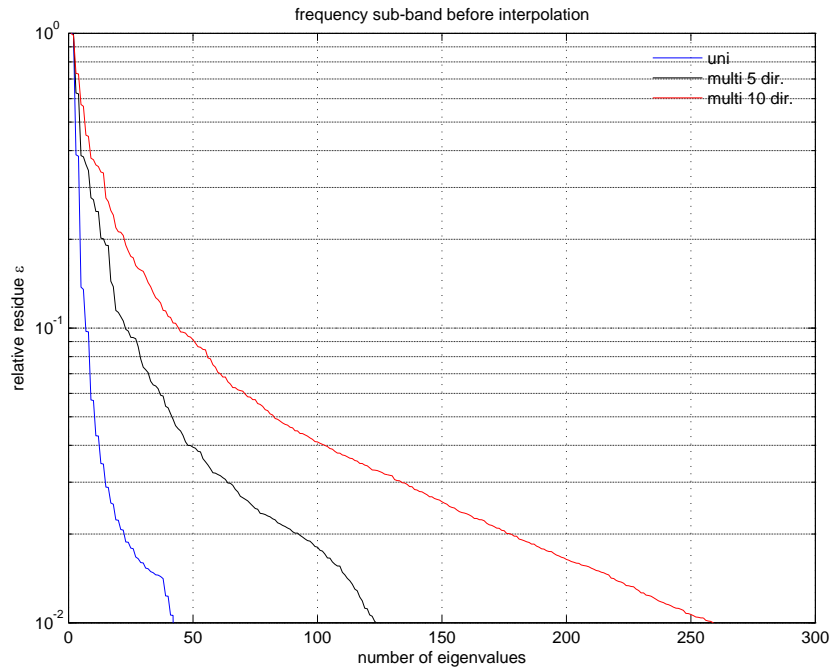
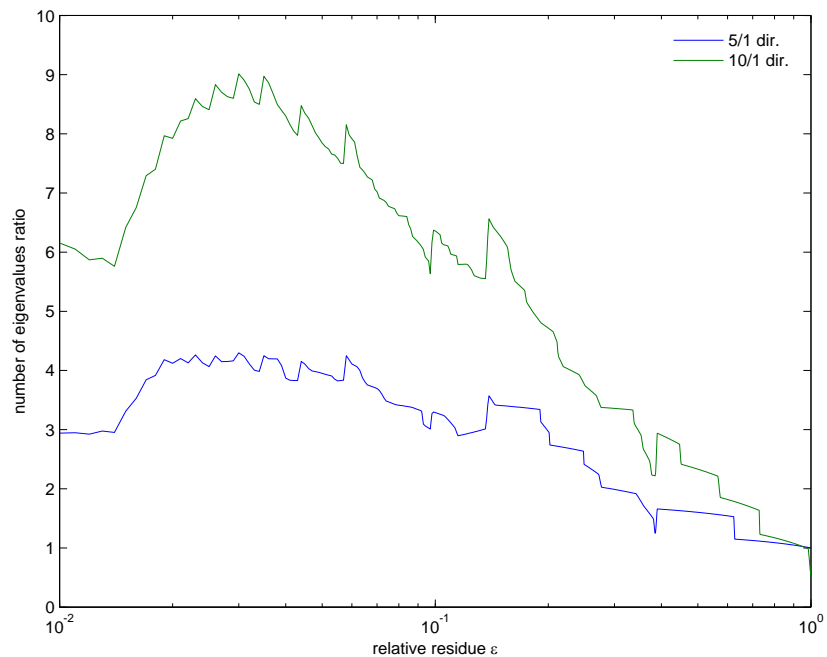


FIGURE 26 : Multi-/uni-directional number of eigenvalues vs relative residue



Conclusion

We have seen in this report, how the approximation of the full matrix of QTF based on eigendecomposition, could permit to reduced the CPU time of the time series reconstruction.

Different options have been discussed (interpolation before/after diagonalisation, use of frequency sub-band, correction of effect of eigenvalue truncation on mean and variance).

The effect of these options and of the number of the eigenvalues on the accuracy of the simulations have been analysed. It showed that the accuracy in average values and extreme statistics stays acceptable with a reduced number of eigenvalues.

Then the CPU time is reduced in such a way that the double summation with the number of waves multiplying the number of waves becomes that with the number of eigenvalues multiplying the number of waves.

In directional QTF case, the gain in CPU time has been estimated. Results obtained in another project on simulation of second order waves by the same technique of approximation of the QTF were more optimistic on the number of eigenvalues for directional case. It seems that it is mainly due to the fact that in the case of second order wave QTF, the QTF is only dependent on the difference between the direction $\theta_i - \theta_j$. This is not the case for the low frequency force QTF, which gives to the QTF a more complex structure.

The gain in CPU time seems to stay more or less the same as in uni-directional, i.e. number of waves divided by number of eigenvalues of the uni-directional case, whatever the number of heading directions. This point has to be confirmed and precised in future studies. Meanwhile, in the case we tested, a gain of 20 in the CPU time is reached with very good accuracy on time series reconstruction. This gain is completely dependent on the density of the QTF around one diagonal.

A mixing of QTF approximation by eigendecomposition and diagonal dominant techniques could certainly permit to reduce more the CPU time, but this needs complementary tedious works.

References

- [1] Karlsen, H.C., Naess, A., 2005, "Statistical response predictions for a nonlinearly moored large volume structure in random seas", *Applied Ocean Research*, vol. 27, no. 6, pp. 273-280.
- [2] Langley, R.S., 1987, "A statistical analysis of non-linear random waves", *Ocean Engineering*, vol. 14, no. 5, pp. 389-407.
- [3] Langley, R.S., 1987, "A statistical analysis of low frequency second-order forces and motions", *Applied Ocean Research*, vol. no. 3, pp. 163-170.
- [4] Lehoucq, R.B., Sorensen, D.C., Yang, C., 1998, "ARPACK Users Guide: Solution of Large-Scale Eigenvalue Problems with Implicitly Restarted Arnoldi Methods". *SIAM*, ISBN-13: 978-0-898714-07-4, p. 160.
- [5] Naess, A., 1996, "A second-order theory for the response statistics of wave induced ship hull vibrations in random seas", *Marine Structures*, vol. 9, no. 3-4, pp. 389-408.
- [6] Radke, R.J., 1996, "Matlab Implementation of the Arnoldi-Lanczos Method for Large-Scale Sparse Eigenvalue Computation", *Masters Thesis*, Dept. Computational and Applied Mathematics, Rice University, Houston, TX.
- [7] Schetzen, M., 1980, "The Volterra&Wiener theories of nonlinear systems", *Wiley-Interscience*.

Appendix 1: Mean, variance and quantile comparisons

In the following tables are gathered all the comparisons between reference time series and approximated ones in the uni-directional case.

- $\varepsilon = \|\text{QTF}_{\text{xev}} - \text{QTF}_{50\text{ev}}\| / \|\text{QTF}_{50\text{ev}}\|$ is the relative Euclidian norm of the residue of approximation of the QTF, when keeping only x eigenvalues.
- $\varepsilon' = \|\text{Q}_{\text{xev}} - \text{Q}_{50\text{ev}}\| / \|\text{Q}_{50\text{ev}}\|$ is the same but on the QTF which includes the wave density spectrum $\text{QTF}(\omega_1, \omega_2)S(\omega_1)S(\omega_2)$.
- x, x' and x'' give the quantiles 0.999, 0.99 and 0.9.
- mean_F_{xev} (resp. mean_S_{xev}) is the mean of forces (resp. displacement) when keeping only x eigenvalues.
- Mf (resp. Ms) is the relative error on the mean of forces (resp. displacements) (Equation (34)).
- Ec_F_{xev} (resp. Ec_S_{xev}) is the standard deviation of forces (resp. displacement) when keeping only x eigenvalues.
- Ef (resp. Es) is the relative error on the standard deviation of forces (resp. displacements) (Equation (35)).
- "inter/diag" (resp. "diag/interp") indicates the order of operations of interpolation and diagonalisation.
- "corrected" indicates that correction on mean and variance have been applied (Equations (32)).

TABLE 4 : 25 eigenvalues

25 eigenvalues

$$\varepsilon = \|\text{QTF}_{25\text{ev}} - \text{QTF}_{50\text{ev}}\|_2 / \|\text{QTF}_{50\text{ev}}\|_2$$

Frequency sub-band :

$$\varepsilon \text{ interp/diag} = 8.13 \cdot 10^{-3}$$

$$\varepsilon \text{ diag/interp} = 1.21 \cdot 10^{-2}$$

All frequencies :

$$\varepsilon \text{ interp/diag} = 3.76 \cdot 10^{-2}$$

$$\varepsilon \text{ diag/interp} = 3.90 \cdot 10^{-2}$$

$$\varepsilon' = \|\text{Q}_{25\text{ev}} - \text{Q}\|_2 / \|\text{Q}\|_2$$

Frequency sub-band :

$$\varepsilon' \text{ interp/diag} = 1.36 \cdot 10^{-2}$$

$$\varepsilon' \text{ diag/interp} = 1.89 \cdot 10^{-2}$$

All frequencies :

$$\varepsilon' \text{ interp/diag} = 1.18 \cdot 10^{-1}$$

$$\varepsilon' \text{ diag/interp} = 1.20 \cdot 10^{-1}$$

Frequency sub-band

	P(F _{25ev} .min> x)=0.999 P(F _{25ev} .min> x')=0.99 P(F _{25ev} .min> x'')=0.90	P(F _{50ev} .min> x)=0.999 P(F _{50ev} .min> x')=0.99 P(F _{50ev} .min> x'')=0.90	Mean	Standard deviation
interp/diag	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.52 10 ⁶	x = -4.00 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F _{25ev} = -2.80 10 ⁵ mean_S _{25ev} = -2.80 10 ⁻¹ Mf = -7.86 10 ⁻³ Ms = -7.86 10 ⁻³	EC_F _{25ev} = 3.26 10 ⁵ EC_S _{25ev} = 7.97 10 ⁻¹ Ef = 1.05 10 ⁻² Es = 5.2 10 ⁻³
diag/interp	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.53 10 ⁶	x = -3.99 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F _{25ev} = -2.85 10 ⁵ mean_S _{25ev} = -2.85 10 ⁻¹ Mf = -2.56 10 ⁻² Ms = -2.56 10 ⁻²	EC_F _{25ev} = 3.27 10 ⁵ EC_S _{25ev} = 7.99 10 ⁻¹ Ef = 1.74 10 ⁻² Es = 9.78 10 ⁻³
interp/diag corrected	x = -4.00 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	x = -4.00 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F = -2.78 10 ⁵ mean_S = -2.78 10 ⁻¹ Mf = -4.51 10 ⁻⁵ Ms = -4.53 10 ⁻⁵	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 1.05 10 ⁻² Es = 5.19 10 ⁻³
diag/interp corrected	x = -3.99 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	x = -3.99e ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F = -2.78 10 ⁵ mean_S = -2.78 10 ⁻¹ Mf = -1.13 10 ⁻⁴ Ms = -9.87 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 1.73 10 ⁻² Es = 9.74 10 ⁻³

All frequencies

	P(F _{25ev} .min> x)=0.999 P(F _{25ev} .min> x')=0.99 P(F _{25ev} .min> x'')=0.90	P(F _{50ev} .min> x)=0.999 P(F _{50ev} .min> x')=0.99 P(F _{50ev} .min> x'')=0.90	Mean	Standard deviation
interp/diag	x = -3.99 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.52 10 ⁶	x = -4.00 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	mean_F _{25ev} = -2.87 10 ⁵ mean_S _{25ev} = -2.87 10 ⁻¹ Mf = -1.90 10 ⁻² Ms = -1.90 10 ⁻²	EC_F _{25ev} = 3.26 10 ⁵ EC_S _{25ev} = 7.97 10 ⁻¹ Ef = 3.07 10 ⁻² Es = 1.67 10 ⁻²
diag/interp	x = -3.97 10 ⁶ x' = -3.28 10 ⁶ x'' = -2.51 10 ⁶	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.53 10 ⁶	mean_F _{25ev} = -2.79 10 ⁵ mean_S _{25ev} = -2.79 10 ⁻¹ Mf = -4.06 10 ⁻² Ms = -4.06 10 ⁻²	EC_F _{25ev} = 3.26 10 ⁵ EC_S _{25ev} = 8.04 10 ⁻¹ Ef = 4.12 10 ⁻² Es = 2.81 10 ⁻²
interp/diag corrected	x = -4.00 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	x = -4.00 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	mean_F = -2.89 10 ⁵ mean_S = -2.89 10 ⁻¹ Mf = -7.48 10 ⁻⁵ Ms = -7.47 10 ⁻⁵	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 3.07 10 ⁻² Es = 1.66 10 ⁻²
diag/interp corrected	x = -4.00 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.53 10 ⁶	mean_F = -2.89 10 ⁵ mean_S = -2.89 10 ⁻¹ Mf = -2.80 10 ⁻⁴ Ms = -3.38 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 4.11 10 ⁻² Es = 2.75 10 ⁻²

TABLE 5 : 10 eigenvalues

10 eigenvalues

$$\varepsilon = \|\text{QTF}_{10\text{ev}} - \text{QTF}_{50\text{ev}}\|_2 / \|\text{QTF}_{50\text{ev}}\|_2$$

Frequency sub-band :

$$\varepsilon \text{ interp/diag} = 3.92 \cdot 10^{-2}$$

$$\varepsilon \text{ diag/interp} = 4.01 \cdot 10^{-2}$$

$$\varepsilon' = \|\text{Q}_{10\text{ev}} - \text{Q}\|_2 / \|\text{Q}\|_2$$

Frequency sub-band :

$$\varepsilon' \text{ interp/diag} = 1.36 \cdot 10^{-2}$$

$$\varepsilon' \text{ diag/interp} = 1.89 \cdot 10^{-2}$$

All frequencies :

$$\varepsilon \text{ interp/diag} = 1.20 \cdot 10^{-1}$$

$$\varepsilon \text{ diag/interp} = 1.00 \cdot 10^{-1}$$

All frequencies :

$$\varepsilon' \text{ interp/diag} = 1.18 \cdot 10^{-1}$$

$$\varepsilon' \text{ diag/interp} = 1.20 \cdot 10^{-1}$$

Frequency sub-band

	P(F _{10ev} .min> x)=0.999 P(F _{10ev} .min> x')=0.99 P(F _{10ev} .min> x'')=0.90	P(F _{50ev} .min> x)=0.999 P(F _{50ev} .min> x')=0.99 P(F _{50ev} .min> x'')=0.90	Mean	Standard deviation
interp/diag	x = -4.01 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	x = -4.00 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F _{10ev} = -2.85 10 ⁵ mean_S _{10ev} = -2.85 10 ⁻¹ Mf = -2.48 10 ⁻² Ms = -2.48 10 ⁻²	EC_F _{10ev} = 3.27 10 ⁵ EC_S _{10ev} = 7.97 10 ⁻¹ Ef = 3.05 10 ⁻² Es = 1.69 10 ⁻²
diag/interp	x = -4.01 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	x = -3.99 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F _{10ev} = -2.85 10 ⁵ mean_S _{10ev} = -2.85 10 ⁻¹ Mf = -2.64 10 ⁻² Ms = -2.64 10 ⁻²	EC_F _{10ev} = 3.27 10 ⁵ EC_S _{10ev} = 7.94 10 ⁻¹ Ef = 3.41 10 ⁻² Es = 2.13 10 ⁻²
interp/diag corrected	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.52 10 ⁶	x = -4.00 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F = -2.78 10 ⁵ mean_S = -2.78 10 ⁻¹ Mf = -2.02 10 ⁻⁴ Ms = -2.12 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 3.05 10 ⁻² Es = 1.69 10 ⁻²
diag/interp corrected	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.52 10 ⁶	x = -3.99 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F = -2.78 10 ⁵ mean_S = -2.78 10 ⁻¹ Mf = -1.91 10 ⁻⁴ Ms = -2.29 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 3.41 10 ⁻² Es = 2.10 10 ⁻²

All frequencies

	P(F _{10ev} .min> x)=0.999 P(F _{10ev} .min> x')=0.99 P(F _{10ev} .min> x'')=0.90	P(F _{50ev} .min> x)=0.999 P(F _{50ev} .min> x')=0.99 P(F _{50ev} .min> x'')=0.90	mean	Standard deviation
interp/diag	x = -3.99 10 ⁶ x' = -3.28 10 ⁶ x'' = -2.53 10 ⁶	x = -4.00 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	mean_F _{10ev} = -3.03 10 ⁵ mean_S _{10ev} = -3.03 10 ⁻¹ Mf = -3.99 10 ⁻² Ms = -3.99 10 ⁻²	EC_F _{10ev} = 3.19 10 ⁵ EC_S _{10ev} = 7.55 10 ⁻¹ Ef = 1.25 10 ⁻¹ Es = 1.08 10 ⁻¹
diag/interp	x = -3.98 10 ⁶ x' = -3.27 10 ⁶ x'' = -2.52 10 ⁶	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.53 10 ⁶	mean_F _{10ev} = -3.03 10 ⁵ mean_S _{10ev} = -3.03 10 ⁻¹ Mf = -4.48 10 ⁻² Ms = -4.48 10 ⁻²	EC_F _{10ev} = 3.18 10 ⁵ EC_S _{10ev} = 7.51 10 ⁻¹ Ef = 1.29 10 ⁻¹ Es = 1.15 10 ⁻¹
interp/diag corrected	x = -4.05 10 ⁶ x' = -3.34 10 ⁶ x'' = -2.56 10 ⁶	x = -4.00 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	mean_F = -2.89 10 ⁵ mean_S = -2.89 10 ⁻¹ Mf = -5.24 10 ⁻⁴ Ms = -8.00 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 1.25 10 ⁻¹ Es = 9.49 10 ⁻²
diag/interp corrected	x = -4.06 10 ⁶ x' = -3.34 10 ⁶ x'' = -2.56 10 ⁶	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.53 10 ⁶	mean_F = -2.89 10 ⁵ mean_S = -2.89 10 ⁻¹ Mf = -6.08 10 ⁻⁴ Ms = -9.03 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 1.28 10 ⁻¹ Es = 9.99 10 ⁻²

TABLE 6 : 5 eigenvalues

5 eigenvalues

$$\varepsilon = \|\text{QTF}_{5ev} - \text{QTF}_{50ev}\|_2 / \|\text{QTF}_{50ev}\|_2$$

Frequency sub-band :

$$\varepsilon \text{ interp/diag} = 1.33 \cdot 10^{-1}$$

$$\varepsilon \text{ diag/interp} = 1.32 \cdot 10^{-1}$$

All frequencies :

$$\varepsilon \text{ interp/diag} = 4.24 \cdot 10^{-1}$$

$$\varepsilon \text{ diag/interp} = 3.13 \cdot 10^{-1}$$

$$\varepsilon' = \|\text{Q}_{5ev} - \text{Q}\|_2 / \|\text{Q}\|_2$$

Frequency sub-band :

$$\varepsilon' \text{ interp/diag} = 4.30 \cdot 10^{-2}$$

$$\varepsilon' \text{ diag/interp} = 1.26 \cdot 10^{-1}$$

All frequencies :

$$\varepsilon' \text{ interp/diag} = 2.44 \cdot 10^{-1}$$

$$\varepsilon' \text{ diag/interp} = 2.50 \cdot 10^{-1}$$

Frequency sub-band

	P(F _{5ev} .min> x)=0.999 P(F _{5ev} .min> x')=0.99 P(F _{5ev} .min> x'')=0.90	P(F _{50ev} .min> x)=0.999 P(F _{50ev} .min> x')=0.99 P(F _{50ev} .min> x'')=0.90	Mean	Standard deviation
interp/diag	x = -3.93 10 ⁶ x' = -3.23 10 ⁶ x'' = -2.47 10 ⁶	x = -4.00 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F _{5ev} = -2.57 10 ⁵ mean_S _{5ev} = -2.57 10 ⁻¹ Mf = 7.50 10 ⁻² Ms = 7.50 10 ⁻²	EC_F _{5ev} = 3.22 10 ⁵ EC_S _{5ev} = 8.04 10 ⁻¹ Ef = 6.07 10 ⁻² Es = 3.40 10 ⁻²
diag/interp	x = -3.98 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.53 10 ⁶	x = -3.99 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F _{5ev} = -2.99 10 ⁵ mean_S _{5ev} = -2.99 10 ⁻¹ Mf = -7.71 10 ⁻² Ms = -7.71 10 ⁻²	EC_F _{5ev} = 3.21 10 ⁵ EC_S _{5ev} = 7.57 10 ⁻¹ Ef = 1.34 10 ⁻¹ Es = 1.10 10 ⁻¹
interp/diag corrected	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.52 10 ⁶	x = -4.00 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F = -2.78 10 ⁵ mean_S = -2.78 10 ⁻¹ Mf = 4.34 10 ⁻⁴ Ms = 5.85 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 5.96 10 ⁻² Es = 3.27 10 ⁻²
diag/interp corrected	x = -4.02 10 ⁶ x' = -3.33 10 ⁶ x'' = -2.54 10 ⁶	x = -3.99 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F = -2.78 10 ⁵ mean_S = -2.78 10 ⁻¹ Mf = -4.45 10 ⁻⁴ Ms = -7.11 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 1.34 10 ⁻¹ Es = 1.00 10 ⁻¹

All frequencies

	P(F _{5ev} .min> x)=0.999 P(F _{5ev} .min> x')=0.99 P(F _{5ev} .min> x'')=0.90	P(F _{50ev} .min> x)=0.999 P(F _{50ev} .min> x')=0.99 P(F _{50ev} .min> x'')=0.90	Mean	Standard deviation
interp/diag	x = -3.75 10 ⁶ x' = -3.08 10 ⁶ x'' = -2.37 10 ⁶	x = -4.00 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	mean_F _{5ev} = -2.79 10 ⁵ mean_S _{5ev} = -2.79 10 ⁻¹ Mf = 4.48 10 ⁻² Ms = 4.48 10 ⁻²	EC_F _{5ev} = 2.89 10 ⁵ EC_S _{5ev} = 6.20 10 ⁻¹ Ef = 2.85 10 ⁻¹ Es = 2.81 10 ⁻¹
diag/interp	x = -3.73 10 ⁶ x' = -3.07 10 ⁶ x'' = -2.36 10 ⁶	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.53 10 ⁶	mean_F _{5ev} = -2.79 10 ⁵ mean_S _{5ev} = -2.79 10 ⁻¹ Mf = 3.98 10 ⁻² Ms = 3.98 10 ⁻²	EC_F _{5ev} = 2.88 10 ⁵ EC_S _{5ev} = 6.15 10 ⁻¹ Ef = 2.92 10 ⁻¹ Es = 2.91 10 ⁻¹
interp/diag corrected	x = -4.21 10 ⁶ x' = -3.45 10 ⁶ x'' = -2.65 10 ⁶	x = -4.00 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	mean_F = -2.89 10 ⁵ mean_S = -2.89 10 ⁻¹ Mf = -1.43 10 ⁻³ Ms = -2.64 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 2.76 10 ⁻¹ Es = 1.85 10 ⁻¹
diag/interp corrected	x = -4.21 10 ⁶ x' = -3.45 10 ⁶ x'' = -2.65 10 ⁶	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.53 10 ⁶	mean_F = -2.78 10 ⁵ mean_S = -2.78 10 ⁻¹ Mf = -1.53 10 ⁻³ Ms = -2.78 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 2.83 10 ⁻¹ Es = 1.95 10 ⁻²

TABLE 7 : 2 eigenvalues
2 eigenvalues

$$\varepsilon = \|\text{QTF}_{2\text{ev}} - \text{QTF}_{50\text{ev}}\|_2 / \|\text{QTF}_{50\text{ev}}\|_2$$

Frequency sub-band :

$$\varepsilon \text{ interp/diag} = 5.36 \cdot 10^{-1}$$

$$\varepsilon \text{ diag/interp} = 5.43 \cdot 10^{-1}$$

All frequencies :

$$\varepsilon \text{ interp/diag} = 9.49 \cdot 10^{-1}$$

$$\varepsilon \text{ diag/interp} = 6.39 \cdot 10^{-1}$$

$$\varepsilon' = \|\text{Q}_{2\text{ev}} - \text{Q}\|_2 / \|\text{Q}\|_2$$

Frequency sub-band :

$$\varepsilon' \text{ interp/diag} = 1.85 \cdot 10^{-1}$$

$$\varepsilon' \text{ diag/interp} = 2.62 \cdot 10^{-1}$$

All frequencies :

$$\varepsilon' \text{ interp/diag} = 9.93 \cdot 10^{-1}$$

$$\varepsilon' \text{ diag/interp} = 9.93 \cdot 10^{-1}$$

Frequency sub-band

	P(F _{2ev} .min> x)=0.999 P(F _{2ev} .min> x')=0.99 P(F _{2ev} .min> x'')=0.90	P(F _{50ev} .min> x)=0.999 P(F _{50ev} .min> x')=0.99 P(F _{50ev} .min> x'')=0.90	Mean	Standard deviation
interp/diag	x = -3.63 10 ⁶ x' = -2.96 10 ⁶ x'' = -2.26 10 ⁶	x = -4.00 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F _{2ev} = -2.03 10 ⁵ mean_S _{2ev} = -2.03 10 ⁻¹ Mf = 2.71 10 ⁻¹ Ms = 2.71 10 ⁻¹	EC_F _{2ev} = 2.94 10 ⁵ EC_S _{2ev} = 7.4 10 ⁻¹ Ef = 2.16 10 ⁻¹ Es = 1.94 10 ⁻²
diag/interp	x = -3.34 10 ⁶ x' = -2.72 10 ⁶ x'' = -2.08 10 ⁶	x = -3.99 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F _{2ev} = -1.88 10 ⁵ mean_S _{2ev} = -1.88 10 ⁻¹ Mf = 3.25 10 ⁻¹ Ms = 3.25 10 ⁻¹	EC_F _{2ev} = 2.69 10 ⁵ EC_S _{2ev} = 6.52 10 ⁻¹ Ef = 2.88 10 ⁻¹ Es = 2.82 10 ⁻¹
interp/diag corrected	x = -4.08 10 ⁶ x' = -3.34 10 ⁶ x'' = -2.56 10 ⁶	x = -4.00 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F = -2.78 10 ⁵ mean_S = -2.78 10 ⁻¹ Mf = 4.00 10 ⁻⁴ Ms = 4.19 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 2.01 10 ⁻¹ Es = 1.77 10 ⁻¹
diag/interp corrected	x = -4.10 10 ⁶ x' = -3.34 10 ⁶ x'' = -2.57 10 ⁶	x = -3.99 10 ⁶ x' = -3.29 10 ⁶ x'' = -2.52 10 ⁶	mean_F = -2.78 10 ⁵ mean_S = -2.78 10 ⁻¹ Mf = 3.14 10 ⁻⁴ Ms = 2.55 10 ⁻⁴	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 2.50 10 ⁻¹ Es = 2.37 10 ⁻¹

All frequencies

	P(F _{2ev} .min> x)=0.999 P(F _{2ev} .min> x')=0.99 P(F _{2ev} .min> x'')=0.90	P(F _{50ev} .min> x)=0.999 P(F _{50ev} .min> x')=0.99 P(F _{50ev} .min> x'')=0.90	Mean	Standard deviation
interp/diag	x = -5.95 10 ⁴ x' = -4.91 10 ⁴ x'' = -3.76 10 ⁴	x = -4.00 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	mean_F _{2ev} = -2.23 10 ³ mean_S _{2ev} = -2.23 10 ⁻³ Mf = 1.01 Ms = 1.01	EC_F _{2ev} = 5.17 10 ³ EC_S _{2ev} = 1.20 10 ⁻² Ef = 1.00 Es = 1.01
diag/interp	x = -5.61 10 ⁴ x' = -4.61 10 ⁴ x'' = -3.54 10 ⁴	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.53 10 ⁶	mean_F _{2ev} = -2.06 10 ³ mean_S _{2ev} = -2.06 10 ⁻³ Mf = 1.00 Ms = 1.00	EC_F _{2ev} = 4.90 10 ³ EC_S _{2ev} = 1.15 10 ⁻² Ef = 1.00 Es = 1.01
interp/diag corrected	x = -3.91 10 ⁶ x' = -3.25 10 ⁶ x'' = -2.52 10 ⁶	x = -4.00 10 ⁶ x' = -3.31 10 ⁶ x'' = -2.53 10 ⁶	mean_F = -2.89 10 ⁵ mean_S = -2.89 10 ⁻¹ Mf = 6.35 10 ⁻³ Ms = 6.29 10 ⁻³	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 1.02 Es = 1.02
diag/interp corrected	x = -3.90 10 ⁶ x' = -3.23 10 ⁶ x'' = -2.52 10 ⁶	x = -4.00 10 ⁶ x' = -3.30 10 ⁶ x'' = -2.53 10 ⁶	mean_F = -2.78 10 ⁵ mean_S = -2.78 10 ⁻¹ Mf = 6.33 10 ⁻³ Ms = 6.27 10 ⁻³	EC_F = 3.27 10 ⁵ EC_S = 7.97 10 ⁻¹ Ef = 1.02 Es = 1.02

