Supporting Information for "Large scale forces under surface gravity waves at a wavy bottom: a mechanism for the generation of primary microseisms"

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Introduction

Text S1. Modulation factor for the wavenumber over a cosine bottom

For small bottom amplitudes, the non-dimensional coefficient α, is the sum of the two modulation effects listed in the paper: the variation of the local wavenumber, and the

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variation of the bottom pressure amplitude, which itself can be split into a modulation of the surface to bottom transfer function, and a modulation of the surface elevation amplitude, conserving the energy flux. This gives $\alpha = \alpha_1 + (\alpha_2 + \alpha_3)$.

We first estimate the wavenumber, given by Taylor expansion around the depth D_0,

$$k(x) = k_0 - d_0 \cos(k_b x) \frac{\partial k_0}{\partial D_0} + O\left(d_0^2 \frac{\partial^2 k}{\partial D^2}\right). \tag{1}$$

Differentiating the dispersion relation

$$\sigma^2 = gk(x) \tanh\left[k(x)D(x))\right]. \tag{2}$$

gives for a constant σ [Ardhuin and Herbers, 2002, appendix D, eq. (D.10)],

$$2\sigma \frac{\partial \sigma}{\partial D} = g \tanh(kD) \frac{\partial k}{\partial D} + \frac{gk}{\cosh^2(kD)} \left(k + D \frac{\partial k}{\partial D}\right) = 0 \tag{3}$$

which for $D = D_0$ gives,

$$\frac{\partial k_0}{\partial D_0} = \frac{-2k_0^2}{2k_0 D_0 + \sinh(2k_0 D_0)}. \tag{4}$$

Second, we estimate the spectrum of a constant amplitude signal that is modulated in wavenumber. For this we write the phase $S(x)$ using $k(x)$ given by eq. (1),

$$S(x) = \int k(x) \mathrm{d}x \simeq \int \left(k_0 - d_0 \cos(k_b x) \frac{\partial k_0}{\partial D_0}\right) \mathrm{d}x = k_0 x + \epsilon \sin(k_b x) + O\left(d_0^2 \frac{\partial^2 k_0}{\partial D_0^2}\right), \tag{5}$$

where we have used the non-dimensional modulation index ϵ given by

$$\epsilon = -\frac{d_0}{k_b} \frac{\partial k_0}{\partial D_0}. \tag{6}$$

Following Cuyt et al. [2008], the Fourier decomposition of a signal with such a modulated phase is given by the Laurent expansion of $\exp[\mathrm{i} z \sin(k_b x)]$,

$$\cos[k_0 x + \epsilon \sin(k_b x) - \sigma t] = \sum_{n=-\infty}^{\infty} J_n(\epsilon) \cos\left[(k_0 + n k_b) x - \sigma t\right], \tag{7}$$

where $J_n(\epsilon)$ is the n-th order Bessel function of the first kind. When $\epsilon \ll 1$, we may replace the Bessel function by its asymptote $J_{-1}(\epsilon) \simeq -\epsilon/2$. We define

$$\alpha_1 = -\frac{\epsilon}{k_0 d_0}. \tag{8}$$

Considering only the lowest harmonics in (7) and in particular the long wavelength component,

$$\cos[S(x) - \sigma t] \simeq \cos(k_0 x - \sigma t) - \alpha_1 k_0 d_0 \frac{1}{2} \cos(k_0 x - k_b x - \sigma t) + ... \tag{9}$$

Using $k_b \simeq k_0$, the value of α_1 simplifies to

$$\alpha_1 = -\frac{d_0}{k_0^2 d_0} \frac{-2k_0^2}{2k_0 D_0 + \sinh(2k_0 D_0)} = \frac{2}{2k_0 D_0 + \sinh(2k_0 D_0)}. \tag{10}$$

The neglected terms in eq. (5) can also give long wavelength components but only at third order with an amplitude $O(d_0^3 \partial^3 k/\partial D^3/k_b)$. Indeed, the second order term is proportional to $\cos^2(k_b x)$ and thus gives Fourier components $\cos\left[(k_0 + 2nk_b)x - \sigma t\right]$ that cannot produce long wavelengths for $k_b \simeq k_0$ but would contribute for $k_b \simeq k_0/2$ when a wide bottom spectrum is considered.

Text S2. Modulation factors for the bottom pressure amplitude over a cosine bottom

Linear wave theory gives

$$p(z = -D) = \rho_w g \frac{a(x) \cos[S(x) - \sigma t]}{\cosh(k(x) D(x))}. \tag{11}$$

With $y = k_0 D_0$, this expands as

$$\begin{aligned} p(z = -D) &= \rho_w g \frac{a(x) \cos[S(x) - \sigma t]}{\cosh y} \frac{\cosh y}{\cosh\left[k(x) D(x)\right]} && (12) \\ &\simeq \rho_w g \frac{a(x) \cos[S(x) - \sigma t]}{\cosh y} \left[1 - \alpha_2 k_0 d(x)\right], && (13) \end{aligned}$$

with

$$\alpha_2 = -\left(1 + \frac{D_0}{k_0}\frac{\partial k_0}{\partial D_0}\right)\tanh(k_0 D_0), \tag{14}$$

$$= -(\sinh^2(y))/\left[y + \sinh(y)\cosh(y)\right] \tag{15}$$

Similarly, the variation of $a(x)$ given by $C_g(x)a^2(x) = C_{g0}a_0^2$ can be expanded as

$$a(x) \simeq a_0\left[1 - \alpha_3 k_0 d(x)\right] \tag{16}$$

$$\text{with} \quad \alpha_3 = -\frac{0.5}{C_{g0}k_0}\frac{\partial C_{g0}}{\partial D_0}. \tag{17}$$

The linear group velocity $C_g = \sigma/k(0.5 + kD/\sinh(2kD))$ gives

$$\frac{\partial C_g}{\partial D} = \frac{\sigma}{k_0}\left[\frac{1}{\sinh(2kD)}\frac{\partial(kD)}{\partial D} - 2kD\frac{\partial(kD)}{\partial D}\frac{\cosh(2kD)}{\sinh(2kD)^2}\right] - \frac{\partial k}{\partial D}\frac{C_g}{k} \tag{18}$$

$$= \frac{\sigma}{k_0}\frac{\partial(k_0 D_0)}{\partial D_0}\left[\frac{1}{\sinh(2y)} - 2y\frac{\cosh(2y)}{\sinh^2(2y)}\right] - 0.5\frac{\sigma}{k_0^2}\frac{\partial k_0}{\partial D_0}\left[1 + \frac{2y}{\sinh(2y)}\right]. \tag{19}$$

Text S3. Combined modulation factor

We now combine $\cos[S(x) - \sigma t]$ using eq. (9) with the results on the bottom pressure amplitude, giving

$$p(z = -D) \simeq \rho_w g a_0 \cos(k_0 x - \sigma t)/\cosh(k_0 D_0) + p_{\mathrm{ls}} + \cdots \tag{20}$$

with the large scale components containing a phase $[(k_0 - k_b)x - \sigma t]$ given by

$$p_{\mathrm{ls}} = -\left(\alpha_1 + \alpha_2 + \alpha_3\right)\frac{k_0 d_0}{2}\cos(k_0 x - k_b x - \sigma t)\frac{\rho_w g a_0}{\cosh(k_0 D_0)}. \tag{21}$$

For $\epsilon \ll 1$, α is a function of $y = kD$ that is the sum of the three parts

$$\alpha_1 = \frac{2}{2y + \sinh(2y)} = \frac{1}{y + \sinh(y)\cosh(y)}, \tag{22}$$

$$\alpha_2 = -\frac{\sinh^2 y}{y + \sinh y \cosh y}, \tag{23}$$

$$\alpha_3 = -\frac{2\sinh(2y) - 4y\sinh^2 y}{(2y + \sinh(2y))^2} = -\frac{\sinh y \cosh y - y\sinh^2 y}{(y + \sinh y \cosh y)^2}. \tag{24}$$

Putting all three terms with a common denominator gives,

$$\alpha(y) = \frac{y + \sinh(y)\cosh(y) - \left[\sinh(y)\cosh(y) + \cosh(y)\sinh^3(y)\right]}{\left[y + \sinh(y)\cosh(y)\right]^2} = \frac{y - \cosh(y)\sinh^3(y)}{\left[y + \sinh(y)\cosh(y)\right]^2}. \tag{25}$$

Figure S1. (a) example of the surface elevation and bottom pressure field for $D_0 = 35$ m and $f = 0.085$ Hz. (b) same surface and elevation with an exponential attenuation of the energy flux $\exp(nx)$ with $n = 10^{-5}\mathrm{m}^{-1}$. (c) and (d) corresponding spectra of the bottom pressure estimated using the numerical data, or from the theoretical large-scale only given by eq. (13) of the paper.

Figure S2. Matlab code used to check the analytical results