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# Supporting Information for ”Atmospheric infrasound radiation from ocean waves in finite depth: a unified generation theory and application to radiation patterns”

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This document presents all the details of derivations necessary to support the paper ”Atmospheric infrasound radiation from ocean waves in finite depth: a unified generation theory and application to radiation patterns”. It follow Brekhovskikh et al. (1973, hereinafter BGKN73), as much as possible. Because we use the more common convention that the velocity vector is $\mathbf{v} = \nabla\phi$ this leads to changes in signs that are highlighted in red. A notable difference with Waxler & Gilbert (2006, hereinafter WG06) is the non-zero value of $\mathbf{k}{\cdot}\mathbf{k}' + kk'$ and similar terms. Some of these were obtained by WG06 using the divergence equation, but not all of them, which misses the azimuthal dependence of the solution.

For convenience we repeat in table 1 the list of notations from the paper, including a few more symbols that were not used in the paper.

**Table S1.** Notations used in different papers: LH50 stands for (Longuet-Higgins, 1950), BGKN73 stands for (Brekhovskikh et al., 1973), WG06 stands for (Waxler & Gilbert, 2006) and AH13 stands for (Ardhuin & Herbers, 2013).

| quantity | this paper | LH50 | BGKN73 | WG06 | AH13 |
|---|---|---|---|---|---|
| vertical coordinate | $z$ | $-z$ | $z$ | $z$ | $z$ |
| angle relative to vertical | $\theta_a$ or $\theta_w$ | – | $\theta$ | – | – |
| surface elevation | $\zeta$ | $\zeta$ | $\zeta$ | $\xi$ | $\zeta$ |
| azimuth of spectrum | $\varphi$ | $\theta$ | $\varphi$ | $\theta$ | $\theta$ |
| azimuth of acoustic signal | $\theta_2$ | – | $\varphi_a$ | – | – |
| velocity potential | $\phi$ | $-\phi$ | $-\varphi$ | $\phi$ | $\phi$ |
| layer index | $l$ | – | $j$ | $\sigma$ | – |
| sound speed | $\alpha_l$ | $c$ | $c_j$ | $c_\sigma$ | $\alpha$ |
| density ratio | $m$ | – | $m$ | – | – |
| horizontal wavenumber | $\mathbf{K}$ | – | $\mathbf{q}$ | – | $\mathbf{K}$ |
| radian frequency | $\Omega$ | – | $\Omega$ | – | $2\pi f_s$ |
| horizontal wavenumbers | $\mathbf{k}$, $\mathbf{k}'$ | $(-uk, -vk)$ | $\varkappa$, $\varkappa_1$ | $\mathbf{k}$, $\mathbf{q}$ | $\mathbf{k}$, $\mathbf{k}'$ |
| radian frequencies | $\sigma\sigma'$ | $\sigma$ | $\omega(\varkappa)$,$\omega(\varkappa_1)$ | $\omega(\mathbf{k})$,$\omega(\mathbf{q})$ | $\sigma\sigma'$ |
| pressure | $p$ | $p$ | $\rho\mathcal{P}$ | $p$ | $p$ |
| vertical wavenumbers | $\nu_\pm$ , $\mu_\pm$ | – , $\alpha$ | $\lambda_1$,$\lambda_2$ | – | $l_a$, $l$ |
| upward amplification | $g/2\alpha_l$ | $\gamma$ | – | – | – |

## S1 EQUATIONS UP TO EQ. (9) IN BGKN73

We start with the **Euler equation** for a perfect fluid (no viscosity), we then use the compressible form of the Bernoulli Equation

$$\rho_l\big(\frac{\partial \mathbf{v}_l}{\partial t} + (\mathbf{v}_l \cdot \nabla)\mathbf{v}_l\big) = -\nabla p_l - g\rho_l \nabla z \qquad \text{(S1.a)}$$

where $g$ is the acceleration of gravity, the subscript $l$ represents the layer, $\mathbf{v}_l$, $\rho_l$ and $p_l$ are respectively the velocity, the density and the pressure of the considered layer. And the **mass conservation equation** gives.

$$\frac{\partial \rho_l}{\partial t} + \nabla(\rho_l \mathbf{v}_l) = 0 \qquad \text{(S1.b)}$$

Equations (2) to (6) in BGKN73 are respectively :

- the **Equation of state** in the linear approximation in the form :

$$p_l - p_{l0}(0) = \alpha_l^2[\rho_l - \rho_{l0}(0)] \tag{S2}$$

- the **boundary conditions equations** to be respected at the interface $z = \zeta(x, y, t)$, which are both the dynamic and kinematic boundary conditions:

$$p_w = p_a, \quad \mathbf{v}_l \nabla z = \partial\zeta/\partial t \quad \text{at} \quad z = \zeta(x, y, t) \tag{S3}$$

- the **equilibrium density and pressure profiles** for ocean and atmosphere are obtained by putting $\mathbf{v}_l = 0$ and $\zeta = 0$ :

$$\begin{aligned} \rho_{l0}(z) &= \rho_{l0}(0)\text{exp}\left\{-gz/\alpha_l^2\right\}, \\ p_{l0}(z) &= p_{l0} + [\rho_{l0}(z) - \rho_{l0}(0)]\,\alpha_l^2 \\ p_{l0} &= p_{a0}(0) = p_{w0}(0) \end{aligned} \tag{S4}$$

- the **expansion of all quantities in a certain small parameter** $\epsilon$ for $\mathbf{v}_l \neq 0$:

$$\begin{aligned} \rho_l &= \rho_{l0}(z) + \epsilon\rho_{l1} + \epsilon^2\rho_{l2} + ... \\ p_l &= p_{l0}(z) + \epsilon p_{l1} + \epsilon^2 p_{l2} + ... \\ \mathbf{v}_l &= \epsilon\mathbf{v}_{l1} + \epsilon^2\mathbf{v}_{l2} + ... \\ \zeta_l &= \epsilon\zeta_{l1} + \epsilon^2\zeta_{l2} + ... \end{aligned} \tag{S5}$$

- the **relation between** $p_{li}$ **and** $\rho_{li}$, obtained from the precedent equations and a series expansion in $\zeta$ and the definition of the quantity $\mathcal{P}_{li}$:

$$p_{li} = \alpha_l^2\rho_{li} \tag{S6.a}$$

$$\text{and} \quad p_{li} = \rho_{l0}(z)\mathcal{P}_{li} \tag{S6.b}$$

where the subscript $i$ is the order of expansion in $\epsilon$

## S1.1 About Euler's equation in BGKN73

Using the expansion in order of $\epsilon$, eq. (S1.a) can be rewritten as

$$\begin{aligned}(\rho_0 &+ \epsilon\rho_1 + \epsilon^2\rho_2)\left(\frac{\partial\epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2}{\partial t} + (\epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2)\cdot\nabla(\epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2)\right) \\ &= -\nabla(p_0 + \epsilon p_1 + \epsilon^2 p_2) - g(\rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2)\nabla z. \qquad \text{(S7)}\end{aligned}$$

Here the subscript layer $l$ is not written to lighten the equations, because the calculation is the same for ocean and atmosphere. Its truncation at the different orders in wave slope $\epsilon$ gives,

• Order 0,

$$-\nabla p_0 = g\rho_0\nabla z \qquad \text{(S8)}$$

• Order 1,

$$\rho_0\frac{\partial\mathbf{v}_1}{\partial t} + 0 = -\nabla p_1 - g\rho_1\nabla z \qquad \text{(S9)}$$

• Order 2:

$$\rho_0\frac{\partial\mathbf{v}_2}{\partial t} + \rho_1\frac{\partial\mathbf{v}_1}{\partial t} + \rho_0\mathbf{v}_1\nabla\mathbf{v}_1 = -\nabla p_2 - g\rho_2\nabla z \qquad \text{(S10)}$$

### *S1.1.1 Simplifications from relations between $p$ and $\rho$*

Then some simplifications arise from eq. (S6.a) and eq. (S6.b)

• Order 1:

$$\nabla p_1 = \nabla(\rho_0\mathcal{P}_1) = \mathcal{P}_1\nabla(\rho_0) + \rho_0\nabla(\mathcal{P}_1) = \mathcal{P}_1\rho_0\cdot\frac{-g}{\alpha^2}\nabla z + \rho_0\nabla(\mathcal{P}_1)$$

And then, remebering from eq. (S6.a) and eq. (S6.b) that $\mathcal{P}_i = \alpha^2\rho_i/\rho_0$ , it simplifies to:

$$\begin{aligned}\mathcal{P}_1\rho_0\cdot\frac{-g}{\alpha^2}\nabla z + \rho_0\nabla(\mathcal{P}_1) &= \frac{\alpha^2\rho_1}{\rho_0}\cdot\rho_0\cdot\frac{-g}{\alpha^2}\nabla z + \rho_0\nabla(\mathcal{P}_1) \\ &= -g\rho_1\nabla z + \rho_0\nabla(P_1)\end{aligned}$$

Finally, Equation (S7) for order 1 becomes

$$\frac{\partial\mathbf{v}_1}{\partial t} + \nabla\mathcal{P}_1 = 0. \qquad \text{(S11)}$$

• Order 2: We similarly obtain

$$-\nabla p_2 - g\rho_2\nabla z = -\nabla(\rho_0\mathcal{P}_2) - g\rho_2\nabla z = -\rho_0\nabla(\mathcal{P}_2) - \alpha^2\frac{\rho_2}{\rho_0}\nabla(\rho_0) - g\rho_2\nabla z$$
$$= -\rho_0\nabla(\mathcal{P}_2) - \alpha^2\frac{\rho_2}{\rho_0}\cdot\frac{-g}{\alpha^2}\nabla z - g\rho_2\nabla z = -\rho_0\nabla(\mathcal{P}_2).$$

Leading to,

$$\frac{\partial \mathbf{v}_2}{\partial t} + \frac{\rho_1}{\rho_0}\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_1\nabla\mathbf{v}_1 = -\nabla\mathcal{P}_2.$$

Remembering from order 1 that $\dfrac{\partial \mathbf{v}_1}{\partial t} = -\nabla\mathcal{P}_1$ (eq. S11) , one obtains

$$\frac{\partial \mathbf{v}_2}{\partial t} + \nabla\mathcal{P}_2 = -\frac{\rho_1}{\rho_0}\frac{\partial \mathbf{v}_1}{\partial t} - \mathbf{v}_1\nabla\mathbf{v}_1 = \frac{\rho_1\alpha^2}{\rho_0\alpha^2}\nabla\mathcal{P}_1 - \mathbf{v}_1\nabla\mathbf{v}_1$$
$$= \frac{1}{\alpha^2}\mathcal{P}_1\nabla\mathcal{P}_1 - \mathbf{v}_1\nabla\mathbf{v}_1 = \frac{1}{2}\nabla\left(\frac{\mathcal{P}_1^2}{\alpha^2} - \mathbf{v}_1^2\right) + \mathbf{v}_1\times\mathbf{rot}\mathbf{v}_1$$

Finally, Equation (S7) for order 2 becomes

$$\frac{\partial \mathbf{v}_2}{\partial t} + \nabla\mathcal{P}_2 = \frac{1}{2}\nabla\left(\frac{\mathcal{P}_1^2}{\alpha^2} - \mathbf{v}_1^2\right) + \mathbf{v}_1\times\mathbf{rot}\mathbf{v}_1 \qquad \text{(S12)}$$

### *S1.1.2 Simplifications from irrotational velocity field*

As the velocity field is irrotational, it can be expressed as the gradient of a potential velocity $\phi$,

$$\mathbf{v}_i = +\nabla\phi_i. \qquad \text{(S13)}$$

This gives,

$$\nabla\mathcal{P}_1 = -\frac{\partial\nabla\phi_1}{\partial t}$$
$$\nabla\mathcal{P}_2 = -\frac{\partial\nabla\phi_2}{\partial t} + \frac{1}{2}\nabla\left(\frac{\mathcal{P}_1^2}{\alpha^2} - (\nabla\phi_1)^2\right)$$

Which can also be written

$$\mathcal{P}_1 = -\frac{\partial\phi_1}{\partial t} \qquad \text{(S14)}$$
$$\mathcal{P}_2 = -\frac{\partial\phi_2}{\partial t} + \left(\frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla\phi_1)^2}{2}\right). \qquad \text{(S15)}$$

## S1.2 About mass conservation equation in BGKN73 and the acoustic wave equation

The same truncation by orders can be done for the mass conservation equation (S1.b):

• Order 1:

$$\frac{\partial \rho_1}{\partial t} + \nabla(\rho_0 \mathbf{v}_1) = 0$$
$$\iff \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \mathbf{v}_1 + \mathbf{v}_1 \nabla \rho_0 = 0$$
$$\iff \frac{\partial \rho_1}{\partial t} = -\rho_0 \Delta \phi_1 - \nabla \phi_1 \nabla \rho_0 \tag{S16}$$

$\frac{\partial}{\partial t}\big(\rho_0 \cdot \text{S14}\big) - \alpha^2 \cdot \text{S16}$ leads to

$$\frac{\partial \rho_0 \mathcal{P}_1}{\partial t} - \alpha^2 \frac{\partial \rho_1}{\partial t} - \alpha^2 \rho_0 \Delta \phi_1 - \alpha^2 \nabla \phi_1 \nabla \rho_0 = -\frac{\partial^2 \rho_0 \phi_1}{\partial t^2}$$
$$\iff \frac{\partial p_1}{\partial t} - \alpha^2 \frac{\partial \alpha^{-2} p_1}{\partial t} - \alpha^2 \rho_0 \Delta \phi_1 + \alpha^2 \nabla \phi_1 \rho_0 \frac{g}{\alpha^2} \nabla z = -\rho_0 \frac{\partial^2 \phi_1}{\partial t^2}$$
$$\iff -\alpha^2 \Delta \phi_1 + g \frac{\partial \phi_1}{\partial z} + \frac{\partial^2 \phi_1}{\partial t^2} = 0$$
$$\iff \Delta \phi_1 - \frac{g}{\alpha^2} \frac{\partial \phi_1}{\partial z} - \frac{1}{\alpha^2} \frac{\partial^2 \phi_1}{\partial t^2} = 0 \tag{S17}$$

• Order 2:

$$\frac{\partial \rho_2}{\partial t} + \nabla(\rho_0 \mathbf{v}_2 + \rho_1 \mathbf{v}_1) = 0$$
$$\iff \frac{\partial \rho_2}{\partial t} + \rho_0 \nabla \mathbf{v}_2 + \mathbf{v}_2 \nabla \rho_0 + \rho_1 \nabla \mathbf{v}_1 + \mathbf{v}_1 \nabla \rho_1 = 0$$
$$\iff \frac{\partial \rho_2}{\partial t} = -\rho_0 \Delta \phi_2 - \nabla \phi_2 \nabla \rho_0 - \rho_1 \Delta \phi_1 - \nabla \phi_1 \nabla \rho_1 \tag{S18}$$

$\frac{\partial}{\partial t}(\rho_0 \cdot \mathrm{S15}) - \alpha^2 \cdot \mathrm{S18}$ leads to:

$$
\begin{aligned}
&\frac{\partial \rho_0 \mathcal{P}_2}{\partial t} - \alpha^2 \frac{\partial \rho_2}{\partial t} - \alpha^2 \rho_0 \Delta\phi_2 - \alpha^2 \nabla\phi_2 \nabla\rho_0 - \alpha^2 \rho_1 \Delta\phi_1 - \alpha^2 \nabla\phi_1 \nabla\rho_1 = -\frac{\partial^2 \rho_0 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\left(\frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla\phi_1)^2}{2}\right) \\
&\iff \frac{\partial p_2}{\partial t} - \alpha^2 \frac{\partial \alpha^{-2} p_2}{\partial t} - \alpha^2 \rho_0 \Delta\phi_2 + \alpha^2 \rho_0 \frac{g}{\alpha^2}\frac{\partial \phi_2}{\partial z} - p_1 \Delta\phi_1 - \nabla\phi_1 \nabla p_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\left(\frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla\phi_1)^2}{2}\right) \\
&\iff -\alpha^2 \rho_0 \Delta\phi_2 + \alpha^2 \rho_0 \frac{g}{\alpha^2}\frac{\partial \phi_2}{\partial z} - \rho_0 \mathcal{P}_1 \Delta\phi_1 - \mathcal{P}_1 \nabla\phi_1 \nabla\rho_0 - \rho_0 \nabla\phi_1 \nabla\mathcal{P}_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\left(\frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla\phi_1)^2}{2}\right) \\
&\iff -\alpha^2 \rho_0 \Delta\phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} - \rho_0 \mathcal{P}_1 \Delta\phi_1 + \mathcal{P}_1 \rho_0 \frac{g}{\alpha^2}\frac{\partial \phi_1}{\partial z} - \rho_0 \nabla\phi_1 \nabla\mathcal{P}_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\left(\frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla\phi_1)^2}{2}\right) \\
&\iff -\alpha^2 \rho_0 \Delta\phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} - \rho_0 \mathcal{P}_1 \left(\Delta\phi_1 - \frac{g}{\alpha^2}\frac{\partial \phi_1}{\partial z}\right) - \rho_0 \nabla\phi_1 \nabla\mathcal{P}_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\left(\frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla\phi_1)^2}{2}\right) \\
&\iff -\alpha^2 \rho_0 \Delta\phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} - \rho_0 \mathcal{P}_1 \frac{1}{\alpha^2}\frac{\partial^2 \phi_1}{\partial t^2} - \rho_0 \nabla\phi_1 \nabla\mathcal{P}_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\left(\frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla\phi_1)^2}{2}\right) \\
&\iff -\alpha^2 \rho_0 \Delta\phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} + \rho_0 \mathcal{P}_1 \frac{1}{\alpha^2}\frac{\partial \mathcal{P}_1}{\partial t} - \rho_0 \nabla\phi_1 \nabla\mathcal{P}_1 = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\left(\frac{\mathcal{P}_1^2}{2\alpha^2} - \frac{(\nabla\phi_1)^2}{2}\right) \\
&\iff -\alpha^2 \rho_0 \Delta\phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} + \rho_0 \nabla\phi_1 \nabla \frac{\partial \phi_1}{\partial t} = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} - \rho_0 \frac{\partial}{\partial t}\frac{(\nabla\phi_1)^2}{2} \\
&\iff -\alpha^2 \rho_0 \Delta\phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} + \rho_0 \frac{\partial}{\partial t}\frac{(\nabla\phi_1)^2}{2} = -\rho_0 \frac{\partial^2 \phi_2}{\partial t^2} - \rho_0 \frac{\partial}{\partial t}\frac{(\nabla\phi_1)^2}{2} \\
&\iff -\alpha^2 \rho_0 \Delta\phi_2 + \rho_0 g \frac{\partial \phi_2}{\partial z} + \rho_0 \frac{\partial^2 \phi_2}{\partial t^2} = -\rho_0 \frac{\partial}{\partial t}(\nabla\phi_1)^2 \\
&\iff \Delta\phi_2 - \frac{g}{\alpha^2}\frac{\partial \phi_2}{\partial z} - \frac{1}{\alpha^2}\frac{\partial^2 \phi_2}{\partial t^2} = +\frac{1}{\alpha^2}\frac{\partial}{\partial t}(\nabla\phi_1)^2 \qquad \text{(S19)}
\end{aligned}
$$

And we retrieve the acoustic wave equation for both first (S17) and second (S19) orders.

## S1.3 About Boundary conditions

We use the same boundary conditions as in BGKN73 for $z = 0$ for velocity,

$$
-\left.\frac{\partial \phi_1}{\partial z}\right|_{z=0} + \frac{\partial \zeta_1}{\partial t} = 0 \qquad \text{(S20)}
$$

$$
-\left.\frac{\partial \phi_2}{\partial z}\right|_{z=0} + \frac{\partial \zeta_2}{\partial t} = -\left(-\left.\frac{\partial^2 \phi_1}{\partial z^2}\right|_{z=0} \zeta_1 + \left.\nabla\phi_1\right|_{z=0} \nabla\zeta_1\right) \qquad \text{(S21)}
$$

And for pressure,

$$
(\mathcal{P}_{w,1} - m\mathcal{P}_{a,1})_{z=0} - g(1-m)\zeta_1 = 0 \qquad \text{(S22)}
$$

$$
\begin{aligned}
(\mathcal{P}_{w,2} - m\mathcal{P}_{a,2})_{z=0} - g(1-m)\zeta_2 &= -\left(\frac{\partial \mathcal{P}_{w,1}}{\partial z} - m\frac{\partial \mathcal{P}_{a,1}}{\partial z}\right)_0 \zeta_1 \\
&\quad + \frac{g}{\alpha_a^2}(n^2 \mathcal{P}_{w,1} - m\mathcal{P}_{a,1})_0 \zeta_1 - \frac{g^2}{2\alpha_a^2}(n^2 - m^2)\zeta_1^2 \qquad \text{(S23)}
\end{aligned}
$$

Here is a summary of the system of equation that corresponds to eq. (9) in BGKN73

$$\Delta\phi_{l,i} - \frac{g}{\alpha_l^2}\frac{\partial\phi_{l,i}}{\partial z} - \frac{1}{\alpha_l^2}\frac{\partial^2\phi_{l,i}}{\partial t^2} = S_{l,i} \qquad \mathcal{P}_{l,i} = -\frac{\partial\phi_{l,i}}{\partial t} + F_{l,i}$$

$$-\left.\frac{\partial\phi_{l,i}}{\partial z}\right|_{z=0} + \frac{\partial\zeta_i}{\partial t} = Q_{l,i} \qquad (\mathcal{P}_{w,i} - m\mathcal{P}_{a,i})_{z=0} - g(1-m)\zeta_i = R_i$$

where,

$$F_{l,1} = S_{l,1} = Q_{l,1} = R_1 = 0$$

$$F_{l,2} = \frac{\mathcal{P}_{l,1}^2}{2\alpha_l^2} - \frac{(\nabla\phi_{l,1})^2}{2}, \qquad S_{l,2} = +\frac{1}{\alpha_l^2}\frac{\partial}{\partial t}(\nabla\phi_{l,1})^2$$

$$Q_{l,2} = -\nabla\phi_{l,1}|_0\nabla\zeta_1 + \left.\frac{\partial^2\phi_{l,1}}{\partial z^2}\right|_{z=0}\zeta_1$$

$$R_2 = -\left(\frac{\partial\mathcal{P}_{w,1}}{\partial z} - m\frac{\partial\mathcal{P}_{a,1}}{\partial z}\right)_0\zeta_1 + \frac{g}{\alpha_a^2}(n^2\mathcal{P}_{w,1} - m\mathcal{P}_{a,1})_0\zeta_1 - \frac{g^2}{2\alpha_a^2}(n^2 - m^2)\zeta_1^2$$

$$m = \rho_{a,0}/\rho_{w,0}, \qquad n = \alpha_a/\alpha_w, \qquad \delta_a = \left(\frac{g}{\alpha_a^2 k}\right)^{1/2} = \frac{\sigma}{\alpha_a k}, \qquad \delta_w = n\delta_a$$

## S2 SOLVING FOR FIRST ORDER AND EXPRESSING THE SECOND ORDER PROBLEM

### S2.1 First order

From the Fourier transform in horizontal space and time we can take

$$\phi_{l,1} = -\mathrm{i}s\sigma\sum\Phi_{l,1}(z)Z\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-s\sigma t)} \tag{S24}$$

The boundary condition in $z=0$ leads to

$$\Phi_{l,1}(z=0) = 1. \tag{S25}$$

Assuming $\Phi_{l,1}(z) = f_l(z)\mathrm{e}^{\gamma_l z}$ with $\gamma_l = g/2\alpha_l^2$ one obtains,

- for the air

$$\phi_{a,1} = \sum\mathrm{i}\frac{s\sigma}{k_a}\mathrm{e}^{-k_a z}Z\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-s\sigma t)} \tag{S26}$$

with

$$\begin{aligned} k_a &= -\gamma_a + k_{a0} \\ &= -\frac{g}{2\alpha_a^2} + \sqrt{k^2 - \gamma_a^2 + \frac{g\gamma_a}{\alpha_a^2} - \frac{\sigma^2}{\alpha_a^2}} \\ &= -\frac{g}{2\alpha_a^2} + k(1 - \frac{g^2}{4\alpha_a^4 k^2} + \frac{g^2}{2\alpha_a^4 k^2} - \frac{\sigma^2}{k^2\alpha_a^2} \\ &= -k\frac{g}{2k\alpha_a^2} + k\left(1 - \frac{\delta_a^4}{4} + \frac{\delta_a^4}{2} - \delta_a^2\right)^{1/2} \\ &= -k\frac{\delta_a^2}{2} + k\left(1 - \frac{\delta_a^2}{2}\right) \\ &= k\left(1 - \delta_a^2\right) \end{aligned}$$

- for the water :

$$\phi_{w,1} = \sum -\mathrm{i}s\sigma \frac{k_{w0}\cosh\left(k_{w0}(z+h)\right) - \gamma_w \sinh\left(k_{w0}(z+h)\right)}{k_w^2 \sinh\left(k_{w0}h\right)} \mathrm{e}^{\gamma_w z} Z \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - s\sigma t)} \tag{S27}$$

with $k_w^2 = k_{w0}^2 - \gamma_w^2 = k^2(1 - 2\delta_w^2)$

If we consider $\delta_w^2$ to be negligible ($\delta_w = n^2\delta_a^2 \simeq 0.05\delta_a^2$) we obtain :

$$\phi_{w,1} = \sum -\mathrm{i}s\sigma \frac{\cosh\left(k_{w0}(z+h)\right)}{k \sinh\left(k_{w0}h\right)} \mathrm{e}^{\gamma_w z} Z \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - s\sigma t)} \tag{S28}$$

For simplicity, from now on we will write that under the form:

$$\phi_{w,1} = \sum -\mathrm{i}s\sigma f_{w,k}(z) \mathrm{e}^{\gamma_w z} Z \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - s\sigma t)} \tag{S29}$$

## S2.2 Second order

At second order, the effects of waves comes into the pressure and velocity boundary conditions at the interfaces, but also as forcing terms on the right hand side of the wave equation. All these different terms take different forms, in particular for waves in intermediate or shallow water (Ardhuin & Herbers, 2013). In the limit of deep water waves, $kh \gg 1$, and neglecting $\delta_w^2$ terms, all the wave forcing terms can be expressed as a function of $\hat{p}_{2,u}$, defined as

$$\hat{p}_{2,u}(x,y,z) = \rho_w |\nabla \phi_1|^2 = \frac{\rho_w g^2}{s\sigma s'\sigma'} \sum \left(\mathbf{k}\cdot\mathbf{k}' - kk'\right) ZZ' \mathrm{e}^{(k+k')z} \mathrm{e}^{\mathrm{i}\Theta} \quad \text{(S30)}$$

with $\Theta = \mathbf{K}\cdot\mathbf{x} - \Omega t$, $\mathbf{K} = \mathbf{k} + \mathbf{k}'$, and $\Omega = s\sigma + s'\sigma'$. A the surface, $z = 0$, this equivalent pressure, correspond to the pressure that drives microseisms as given by Hasselmann (1963, eq. 2.12).

In the following, we will neglect all the short wavelength components that correspond to the middle line of eq. (2.13) of (Hasselmann, 1963), keeping only the large wavelengths that excite microbaroms, and for which $|\mathbf{k} + \mathbf{k}'| \ll |\mathbf{k}|$.

Given that acoustic waves in the atmosphere are much slower than those in water, we will retain $\delta_a^2$ terms. As a result, following (Brekhovskikh et al., 1973), we cannot use the approximation $\mathbf{k}\cdot\mathbf{k}' \simeq 0$, but instead, using $\mathbf{k}\cdot\mathbf{k}' < 0$ for those components that produce microseisms, we can use

$$\frac{K}{k} = \frac{K\alpha_a}{2\sigma}\frac{2\sigma}{k\alpha_a} = 2\sin\theta_a\delta_a \quad \text{(S31)}$$

and the law of cosine in triangles,

$$2\mathbf{k}\cdot\mathbf{K} = k^2 + K^2 - k'^2 \quad \text{(S32)}$$

this gives,

$$\begin{aligned}
kk' + \mathbf{k}\cdot\mathbf{k}' &= kk'\left[1 - \left(\left(\frac{-\mathbf{k}\cdot\mathbf{k}'}{kk'}\right)^2\right)^{1/2}\right] = kk'\left[1 - \left(\left(\frac{\mathbf{K}\cdot\mathbf{k}' - \mathbf{k}'\cdot\mathbf{k}'}{kk'}\right)\left(\frac{\mathbf{k}\cdot\mathbf{K} - \mathbf{k}\cdot\mathbf{k}}{kk'}\right)\right)^{1/2}\right] \\
&= kk'\left[1 - \left(\left(\frac{-\mathbf{K}\cdot\mathbf{k} + K^2 - k'^2}{kk'}\right)\left(\frac{\mathbf{k}\cdot\mathbf{K} - k^2}{kk'}\right)\right)^{1/2}\right] \\
&= kk'\left[1 - \left(\frac{k^2k'^2 - (\mathbf{K}\cdot\mathbf{k})^2 + \mathbf{K}\cdot\mathbf{k}\left(k^2 + K^2 - k'^2\right) - k^2K^2}{k^2k'^2}\right)^{1/2}\right] \\
&= kk'\left[1 - \left(1 + \frac{-(\mathbf{K}\cdot\mathbf{k})^2 + 2\mathbf{K}\cdot\mathbf{k}(\mathbf{K}\cdot\mathbf{k}) - k^2K^2}{k^2k'^2}\right)^{1/2}\right] \\
&= kk'\left[1 - \left(1 + \frac{(\mathbf{K}\cdot\mathbf{k})^2 - k^2K^2}{k^2k'^2}\right)^{1/2}\right] \\
&\simeq kk'\left[-\frac{1}{2}\left(\frac{(\mathbf{K}\cdot\mathbf{k})^2}{k^2k'^2} - \frac{K^2}{k'^2}\right)\right] \simeq kk'\frac{1}{2}\frac{K^2}{k'^2}\left[1 - \left(\frac{(\mathbf{K}\cdot\mathbf{k})^2}{k^2K^2}\right)\right] \\
&\simeq 2kk'\sin^2\theta_a\delta_a^2\left[1 - \left(\frac{(\mathbf{k}\cdot\mathbf{K})^2}{k^2K^2}\right)\right] = 2kk'\sin^2\theta_a\delta_a^2\left[1 - \cos^2(\varphi_2 - \varphi)\right] \quad \text{(S33)}
\end{aligned}$$

which is a function of the azimuth $\varphi_2$ of the acoustic wave propagation, with $\cos(\varphi_2 - \varphi) =$

$\mathbf{k}{\cdot}\mathbf{K}/(kK)$.

Then,

$$\mathbf{k}{\cdot}\mathbf{k}' - kk' \simeq -2kk'\left(1 - \sin^2\theta_a\delta_a^2\left[1 - \cos^2(\varphi_2 - \varphi)\right]\right) \tag{S34}$$

This gives,

$$\hat{p}_{2,u}(x,y,z) \simeq -2\rho_w\sigma\sigma'\sum\left(1 - \sin^2\theta_a\delta_a^2\left[1 - \cos^2(\varphi_2 - \varphi)\right]\right) ZZ'\mathrm{e}^{(k+k')z}\mathrm{e}^{\mathrm{i}\Theta} \tag{S35}$$

Other similar terms have more simple forms with no azimuthal dependency

$$\frac{1}{2}\left(k^2 + \mathbf{k}{\cdot}\mathbf{k}' + k'^2 + \mathbf{k}{\cdot}\mathbf{k}'\right) = \frac{1}{2}\left(\mathbf{k}{\cdot}\mathbf{K} + \mathbf{k}'{\cdot}\mathbf{K}\right) = \frac{1}{2}K^2 \simeq 2k^2\sin^2\theta_a\delta_a^2. \tag{S36}$$

# S3 SECOND ORDER SOLUTION

## S3.1 General form of the solution in the water layer

The homogeneous solution is obtained for $S_{w,2} = 0$,

$$\phi_{w,2,h}(x,y,z,t) = \sum\Phi_{w,2,h}\mathrm{e}^{\mathrm{i}\Theta}, \quad \text{with} \quad \Theta = \mathbf{K}{\cdot}\mathbf{x} - \Omega t, \quad \mathbf{K} = \mathbf{k} + \mathbf{k}', \quad \Omega = s\sigma + s'\sigma' \tag{S37}$$

Assuming a $\mathrm{e}^{\mathrm{i}\mu z}$ variation over the vertical and replacing eq. (S37) in the homogeneous equation (S19) gives,

$$\mu^2 + \mathrm{i}\frac{g}{\alpha_w^2}\mu + (K^2 - \Omega^2/\alpha_w^2) = 0 \tag{S38}$$

with solutions,

$$\mu_\pm = -\mathrm{i}\frac{g}{\alpha_w^2} \pm \sqrt{\frac{g^2}{2\alpha_w^4} + (\Omega^2/\alpha_w^2 - K^2)} \simeq \pm k_{w2,0}(1 + O(\delta_w^2)) \tag{S39}$$

with the complex wavenumber $k_{w2,0} = \sqrt{\Omega^2/\alpha_w^2 - K^2}$ so that the homogeneous solution is

$$\Phi_{w,2,h} = W_+\mathrm{e}^{\mathrm{i}\mu_+ z} + W_-\mathrm{e}^{\mathrm{i}\mu_- z}. \tag{S40}$$

We recall that the wave equation is forced by,

$$S_{w,2} = +\frac{1}{\alpha_w^2}\frac{\partial}{\partial t}\left(\nabla\phi_1\right)^2 = +\frac{1}{\rho_w\alpha^2}\frac{\partial\widehat{p}_{2,u}}{\partial t} \tag{S41}$$

This forcing adds a particular solution of order $\delta_w^2$ that could be neglected here but we will only keep the lowest order term to be consistent with BGKN73. This is also discussed by (Longuet-Higgins, 1950) and (Waxler & Gilbert, 2006). We will only give its expression in the limit of deep water, i.e. $kh \gg 1$.

We recall the right hand side of eq. eq:Sw2,

$$S_{w,2}(x,z,t) \simeq +\frac{1}{\rho_w \alpha_w^2}\frac{\partial \widehat{p}_{2,u}(x,y,z,t)}{\partial t} = -\frac{g^2}{\alpha_a^2}\sum \mathrm{i}\frac{s\sigma + s\sigma'}{s\sigma s\sigma'}\left(\mathbf{k}\cdot\mathbf{k}' - kk'\right) ZZ' \mathrm{e}^{(k_w+k_w')z}\mathrm{e}^{\mathrm{i}\Theta}. \tag{S42}$$

Looking for a solution of the form

$$\phi_{w,2,p} = \sum \Phi_{w,2,p}\mathrm{e}^{\mathrm{i}\Theta} \tag{S43}$$

We replace it in the wave equation (S19) and find

$$\Phi_{w,2,p} \simeq -\mathrm{i}\frac{g^2}{u}\cdot\frac{s\sigma + s\sigma'}{s\sigma s\sigma'}\left(\mathbf{k}\cdot\mathbf{k}' - kk'\right) ZZ' \mathrm{e}^{(k_w+k_w')z}. \tag{S44}$$

with the denominator defined by

$$u = \alpha_w^2\left[-K^2 + \frac{\Omega^2}{\alpha_w^2} + (k_w + k_w')^2\right] + g(k_w + k_w') \simeq \alpha_w^2(k_w + k_w')^2 \simeq 4\alpha_w^2 k^2. \tag{S45}$$

Of particular interest is the long-wavelength part – with $s = s'$ – of the vertical derivative of $\phi_{w,2,p}$, given by,

$$\begin{aligned}\frac{\partial \phi_{w,2,p}}{\partial z} &\simeq \sum -\mathrm{i}s\frac{g^2}{4\alpha_w^2 k^2}\frac{2\sigma(k_w + k_w')}{\sigma\sigma'}\left(\mathbf{k}\cdot\mathbf{k}' - kk'\right) ZZ' \mathrm{e}^{(k_w+k_w')z}\mathrm{e}^{\mathrm{i}\Theta},\\ &\simeq +\sum \mathrm{i}s\delta_w^2\frac{g}{\sigma}2k^2 ZZ' \mathrm{e}^{(k_w+k_w')z}\mathrm{e}^{\mathrm{i}\Theta}.\end{aligned} \tag{S46}$$

### S3.2 General form of the solution in the air layer

For the air, we only consider acoustic waves radiating upward, giving the homogeneous solution,

$$\phi_{a,2,h}(x,y,z,t) = \sum sA_+ ZZ' \mathrm{e}^{\nu_+ z}\mathrm{e}^{\mathrm{i}\Theta}, \tag{S47}$$

where

$$\nu_+ = \frac{g}{\alpha_a^2} + \mathrm{i}\sqrt{\frac{g^2}{2\alpha^4} + (\Omega^2/\alpha_w^2 - K^2)}. \tag{S48}$$

For the particular solution, we recall the right hand side,

$$S_{a,2}(x,z,t) \simeq +\frac{1}{\rho_w \alpha_a^2}\frac{\partial \widehat{p}_{2,u}(x,y,-z,t)}{\partial t} = -\frac{g^2}{\alpha_a^2}\sum \mathrm{i}\frac{s\sigma + s\sigma'}{s\sigma s\sigma'}\left(\mathbf{k}\cdot\mathbf{k}' - k_a k_a'\right) ZZ' \mathrm{e}^{-(k_a+k_a')z}\mathrm{e}^{\mathrm{i}\Theta}. \tag{S49}$$

Looking for a solution of the form

$$\phi_{a,2,p} = \sum \Phi_{a,2,p} \mathrm{e}^{\mathrm{i}\Theta} \tag{S50}$$

We replace it in the wave equation (S19) and find

$$\Phi_{a,2,p} \simeq -\mathrm{i}\frac{g^2}{u}\frac{s\sigma + s\sigma'}{s\sigma s\sigma'}\left(\mathbf{k}\cdot\mathbf{k}' - k_a k_a'\right) Z Z' \mathrm{e}^{-(k_a+k_a')z}. \tag{S51}$$

with the denominator defined by

$$u = \alpha_a^2\left[-K^2 + \frac{\Omega^2}{\alpha_a^2} + (k_a + k_a')^2\right] + g(k_a + k_a') \simeq \alpha_a^2 (k_a + k_a')^2 \simeq 4\alpha_a^2 k^2. \tag{S52}$$

The derivation of eq. (S52) is detailed below:

$$\begin{aligned}
u &= \alpha_a^2\left[-K^2 + \frac{\Omega^2}{\alpha_a^2} + (k_a + k_a')^2\right] + g(k_a + k_a') \\
&\simeq \alpha_a^2 k^2 \left(4\delta_a^2 \cos^2\theta_a + \left(1 + \frac{k'}{k}\right)^2 (1 - 2\delta_a^2) + \frac{g}{\alpha_a^2 k^2}(k_a + k_a')\right) \\
&\simeq \alpha_a^2 k^2 (4\delta_a^2 \cos^2\theta_a + 4(1 - 2\delta_a^2) + 2\delta_a^2(1 - \delta_a^2)) \\
&\simeq 4\alpha_a^2 k^2 (1 - \delta_a^2(\sin^2\theta_a + \frac{1}{2})) \simeq 4\alpha_a^2 k^2 (1 + O(\delta_a^2))
\end{aligned} \tag{S53}$$

Of particular interest is the long-wavelength part – with $s = s'$ – of the vertical derivative of $\phi_{a,2,p}$, given by,

$$\begin{aligned}
\frac{\partial \phi_{a,2,p}}{\partial z} &\simeq +\sum \mathrm{i}s\frac{g^2}{4\alpha_a^2 k^2}(1 - O(\delta_a^2))\frac{2\sigma(k_a + k_a')}{\sigma\sigma'}\left(\mathbf{k}\cdot\mathbf{k}' - k_a k_a'\right) Z Z' \mathrm{e}^{-(k_a+k_a')z}\mathrm{e}^{\mathrm{i}\Theta}, \\
&\simeq +\sum \mathrm{i}s\frac{g^2}{4\alpha_a^2 k^2}(1 - O(\delta_a^2))\frac{2\sigma(k + k')(1 - \delta_a^2)}{\sigma\sigma'}\left(\mathbf{k}\cdot\mathbf{k}' - kk' - 2kk'\delta_a^2\right) Z Z' \mathrm{e}^{-(k_a+k_a')z}\mathrm{e}^{\mathrm{i}\Theta}, \\
&\simeq +\sum \mathrm{i}s\delta_a^2\frac{g}{\sigma}\left(\mathbf{k}\cdot\mathbf{k}' - kk'\right)(1 + O(\delta_a^2)) Z Z' \mathrm{e}^{-(k_a+k_a')z}\mathrm{e}^{\mathrm{i}\Theta}, \\
&\simeq -\sum \mathrm{i}s\delta_a^2 2\sigma' k Z Z' \mathrm{e}^{-(k_a+k_a')z}\mathrm{e}^{\mathrm{i}\Theta}.
\end{aligned} \tag{S54}$$

## S3.3 The BGKN terms - $F_{l,2}$, $Q_{l,2}$, $R_2$

### *S3.3.1 In the water layer*

To simplify the calculation of these terms we use $kh \gg 1$ for waves in deep water, and $k_{w0} \simeq k$, we may also use eq. (S33) and eq. (S34). These simplifications lead to :

$$\phi_{w,1} = \sum -\mathrm{i}\frac{s\sigma}{k}\mathrm{e}^{k_{w0}z}\mathrm{e}^{\gamma_w z} Z \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - s\sigma t)} \tag{S55}$$

And then we obtain the second order terms :

$$
\begin{aligned}
F_{w,2}(z=0) &= \left.\frac{\mathcal{P}_{w,1}^2}{2\alpha_w^2}\right|_0 - \left.\frac{\left(\nabla\phi_{w,1}\right)^2}{2}\right|_0 \\
&= \sum \frac{ss'\sigma\sigma'}{2kk'}\left[\left(\frac{s\sigma s'\sigma'}{\alpha_w^2} - \mathbf{k}\mathbf{k}' + kk'\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq \sum \sigma^2\left[1 + \delta_a^2\left(\frac{n^2}{2} - \sin^2\theta_a\left[1-\cos^2(\varphi_2-\varphi)\right]\right]\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta}.
\end{aligned}
\tag{S56}
$$

Using the law of cosines in a triangle,

$$
k'^2 = k^2 + K^2 - 2\mathbf{k}\cdot\mathbf{K} \tag{S57}
$$

so that

$$
\sqrt{k'} = \sqrt{k}\left(1 + \frac{K^2 - 2\mathbf{k}\cdot\mathbf{K}}{k^2}\right)^{1/4} \simeq \sqrt{k}\left(1 + \frac{1}{4}\frac{K^2 - 2\mathbf{k}\cdot\mathbf{K}}{k^2}\right) \tag{S58}
$$

we get

$$
\begin{aligned}
Q_{w,2}|_{z=0} &= + \left.\frac{\partial^2\phi_{w,1}}{\partial z^2}\right|_0 \zeta_1 - \nabla\phi_{w,1}|_0\cdot\nabla\zeta_1 \\
&\simeq -\mathrm{i}\sum\left[\frac{s\sigma k + s'\sigma' k'}{2} + \frac{\mathbf{k}\cdot\mathbf{k}'}{2}\left(\frac{s\sigma}{k} + \frac{s'\sigma'}{k'}\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&= -\mathrm{i}\sum\left[s\sigma\frac{1}{2k}\left(k^2 + \mathbf{k}\cdot\mathbf{k}'\right) + s'\frac{\sigma'}{2k'}\left(k'^2 + \mathbf{k}\cdot\mathbf{k}'\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq -\mathrm{i}\sum\left[s\sigma\frac{\mathbf{k}\cdot(\mathbf{k}+\mathbf{k}')}{2k} + s'\sigma'\frac{\mathbf{k}'\cdot(\mathbf{k}'+\mathbf{k})}{2k'}\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq -\mathrm{i}\sum\left[s\sigma\frac{\mathbf{k}\cdot\mathbf{K}}{2k} + s'\sigma'\frac{\mathbf{k}'\cdot\mathbf{K}}{2k'}\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq -\mathrm{i}\sum s\left[\frac{\sigma}{2k}\left(\mathbf{k}\cdot\mathbf{K} + \mathbf{k}'\cdot\mathbf{K}\right) + \frac{\sigma' k - \sigma k'}{2k'k}\left(\mathbf{k}'\cdot\mathbf{K}\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq -\mathrm{i}\sum s\sigma k\left[\frac{K^2}{2k^2} + \sqrt{g}\frac{\sqrt{k}-\sqrt{k'}}{2k\sigma\sqrt{kk'}}\left(-\mathbf{k}\cdot\mathbf{K} + K^2\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq -\mathrm{i}\sum s\sigma k\left[2\delta_a^2\sin^2\theta_a + \frac{1}{4k^2}\left(2\mathbf{k}\cdot\mathbf{K} - K^2\right)\right)\frac{1}{2k^2}\left(-\mathbf{k}\cdot\mathbf{K} + K^2\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq -\mathrm{i}\sum s\sigma k\left[2\delta_a^2\sin^2\theta_a - \frac{K^2}{4k^2}\left(\frac{\mathbf{k}\cdot\mathbf{K}}{kK}\right)^2\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq -\mathrm{i}\sum s\sigma k\left[2\delta_a^2\sin^2\theta_a\left(1 - \frac{1}{2}\cos^2(\varphi_2-\varphi)\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta}.
\end{aligned}
\tag{S59}
$$

### *S3.3.2 In the air*

In a similar way we obtain :

$$
\begin{aligned}
F_{a,2}(z=0) &= \left.\frac{\mathcal{P}_{w,1}^2}{2\alpha_w^2}\right|_0 - \left.\frac{\left(\nabla\phi_{w,1}\right)^2}{2}\right|_0 \\
&= \sum \frac{ss'\sigma\sigma'}{2k_a k_a'}\left[\frac{s\sigma s'\sigma'}{\alpha_a^2} - \mathbf{k}\mathbf{k}' + k_a k_a'\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&= \sum ss'\sigma\sigma'(1+2\delta_a^2)\left[\frac{s\sigma s'\sigma'}{2kk'\alpha_a^2} - \frac{\mathbf{k}\mathbf{k}' - kk' + 2\delta_a^2 kk'}{2kk'}\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq \sum \sigma^2\left[1+\delta_a^2\left(\frac{3}{2} - \sin^2\theta_a\left[1-\cos^2(\varphi_2-\varphi)\right]\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \qquad \text{(S60)}
\end{aligned}
$$

And using :

$$
\begin{aligned}
\frac{s\sigma k + s'\sigma'k'}{2} - \frac{\mathbf{k}\cdot\mathbf{k}'}{2}\left(\frac{s\sigma}{k} + \frac{s'\sigma'}{k'}\right) &= s\sigma\frac{1}{2k}\left(k^2 - \mathbf{k}\cdot\mathbf{k}'\right) + s'\frac{\sigma'}{2k'}\left(k'^2 - \mathbf{k}\cdot\mathbf{k}'\right) \\
&= s\sigma\frac{1}{2k}\left(2k^2 - \mathbf{k}\cdot\mathbf{K}\right) + s'\frac{\sigma'}{2k'}\left(2k'^2 - \mathbf{k}'\cdot\mathbf{K}\right) \\
&= s\sigma k + s'\sigma'k' - \left[s\sigma\frac{\mathbf{k}\cdot\mathbf{K}}{2k} + s'\sigma'\frac{\mathbf{k}'\cdot\mathbf{K}}{2k'}\right] \\
&\simeq s\sigma k\left[2 - 2\delta_a^2\sin^2\theta_a\left(1 - \frac{1}{2}\cos^2(\varphi_2-\varphi)\right)\right] \qquad \text{(S61)}
\end{aligned}
$$

one gets :

$$
\begin{aligned}
Q_{a,2}|_{z=0} &= + \left.\frac{\partial^2\phi_{a,1}}{\partial z^2}\right|_0 \zeta_1 - \nabla\phi_{a,1}|_0\cdot\nabla\zeta_1 \\
&= +\mathrm{i}\sum\left[\frac{s\sigma k_a + s'\sigma'k_a'}{2} + \frac{\mathbf{k}\cdot\mathbf{k}'}{2}\left(\frac{s\sigma}{k_a} + \frac{s'\sigma'}{k_a'}\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&= +\mathrm{i}\sum\left[\frac{s\sigma k + s'\sigma'k'}{2}(1-\delta_a^2) + \frac{\mathbf{k}\cdot\mathbf{k}'}{2}\left(\frac{s\sigma}{k} + \frac{s'\sigma'}{k'}\right)(1+\delta_a^2)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&= +\mathrm{i}\sum\left[\frac{s\sigma k + s'\sigma'k'}{2} + \frac{\mathbf{k}\cdot\mathbf{k}'}{2}\left(\frac{s\sigma}{k}+\frac{s'\sigma'}{k'}\right) - \delta_a^2\cdot\left(\frac{s\sigma k + s'\sigma'k'}{2} - \frac{\mathbf{k}\cdot\mathbf{k}'}{2}\left(\frac{s\sigma}{k}+\frac{s'\sigma'}{k'}\right)\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq +\mathrm{i}\sum s\sigma k 2\delta_a^2\left[\sin^2\theta_a\left(1-\frac{1}{2}\cos^2(\varphi_2-\varphi)\right) - \left[1-\delta_a^2\sin^2\theta_a\left(1-\frac{1}{2}\cos^2(\varphi_2-\varphi)\right)\right]\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
&\simeq -\mathrm{i}\sum s\sigma k 2\delta_a^2\left[1 - \sin^2\theta_a\left(1-\frac{1}{2}\cos^2(\varphi_2-\varphi)\right)(1+\delta_a^2)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \qquad \text{(S62)}
\end{aligned}
$$

#### S3.3.3 $R_2$ coefficient

$$\begin{aligned}
R_2 = & -\left(\frac{\partial \mathcal{P}_{w,1}}{\partial z} - m\frac{\partial \mathcal{P}_{a,1}}{\partial z}\right)_0 \zeta_1 + \frac{g}{\alpha_a^2}(n^2\mathcal{P}_{w,1} - m\mathcal{P}_{a,1})_0\zeta_1 - \frac{g^2}{2\alpha_a^2}(n^2-m^2)\zeta_1^2 \\
= & \sum\left[\frac{\sigma^2+\sigma'^2}{2}\cdot(1-m) - \delta_a^2\frac{\sigma^2+\sigma'^2}{2}(n^2+m(1+\delta_a^2)) - \frac{\delta_a^2}{2}\frac{\sigma^2+\sigma'^2}{2}(n^2-m^2)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \\
\simeq & \sum\left[\sigma^2\cdot\left(1-m-\delta_a^2\left(\frac{3n^2}{2}+m\left(1-\frac{m}{2}+\delta_a^2\right)\right)\right)\right] ZZ'\mathrm{e}^{\mathrm{i}\Theta} \qquad \text{(S63)}
\end{aligned}$$

## S4 MATRIX PROBLEM FOR THE SECOND ORDER AMPLITUDES

• Velocity continuity at $z = 0$

$$\left.\frac{\partial \phi_{a,2}}{\partial z}\right|_0 + Q_{a,2} = \left.\frac{\partial \phi_{w,2}}{\partial z}\right|_0 + Q_{w,2} \tag{S64}$$

$$\iff \nu_+ A_+ - \mu_- W_- - \mu_+ W_+ = \left.\frac{\partial \Phi_{w,2,p}}{\partial z}\right|_0 - \left.\frac{\partial \Phi_{a,2,p}}{\partial z}\right|_0 - Q_{a,2} + Q_{w,2} \tag{S65}$$

• Pressure continuity at $z = 0$

$$\left(\frac{\partial \mathcal{P}_{w,2}}{\partial t} - m\frac{\partial \mathcal{P}_{a,2}}{\partial t}\right)_0 - g(1-m)\frac{\partial \zeta_2}{\partial t} = \frac{\partial R_2}{\partial t}$$

$$\iff \left(\frac{\partial \mathcal{P}_{w,2}}{\partial t} - m\frac{\partial \mathcal{P}_{a,2}}{\partial t}\right)_0 - g(1-m)\left.\frac{\partial \phi_{a,2}}{\partial z}\right|_0 - g(1-m)Q_{a,2} = \frac{\partial R_2}{\partial t}$$

$$\iff -\left.\frac{\partial^2 \phi_{w,2}}{\partial t^2}\right|_0 + \left.\frac{\partial F_{w,2}}{\partial t}\right|_0 + m\left.\frac{\partial^2 \phi_{a,2}}{\partial t^2}\right|_0 - m\left.\frac{\partial F_{a,2}}{\partial t}\right|_0 - g(1-m)\left.\frac{\partial \phi_{a,2}}{\partial z}\right|_0 - g(1-m)Q_{a,2} = \frac{\partial R_2}{\partial t}$$

$$\iff -\Omega^2\left(-\phi_{w,2,p}(0) - W_+ - W_- + m\phi_{a,2,p}(0) + mA_+\right) + \mathrm{i}\Omega(mF_{a,2}(0) - F_{w,2}(0))$$

$$-g(1-m)(\phi'_{a,2,p}(0) + \nu A_+ + Q_{a,2}) = \frac{\partial R_2}{\partial t} \tag{S66}$$

• Boundary condition at $z = -h$

This boundary condition is given as an example below. A more realistic boundary condition will be developed further:

$$\frac{\partial \Phi_{w,2,p}}{\partial z}(-h) + \mu_- W_- \mathrm{e}^{-\mu_- h} + \mu_+ W_2 \mathrm{e}^{-\mu_+ h} = 0 \tag{S67}$$

Then we can write the boundary conditions system as a matrix problem,

$$\begin{pmatrix} \nu & -\mu_- & -\mu_+ \\ -m\Omega^2 - g(1-m)\nu & \Omega^2 & \Omega^2 \\ 0 & \mu_-\mathrm{e}^{-\mu_- h} & \mu_+\mathrm{e}^{-\mu_+ h} \end{pmatrix} \cdot \begin{pmatrix} A_+ \\ W_- \\ W_+ \end{pmatrix} = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{pmatrix} \tag{S68}$$

Because we have assumed $kh \gg 1$ we can neglect the $p_{bot}$ term of (Ardhuin & Herbers, 2013) in $\Lambda_3$, and the $\Lambda$ forcing terms are,

$$\Lambda_1 = \left.\frac{\partial \phi_{w,2,p}}{\partial z}\right|_0 - \left.\frac{\partial \phi_{a,2,p}}{\partial z}\right|_0 - Q_{a,2} + Q_{w,2} \tag{S69}$$

$$\Lambda_2 = -\Omega^2\big(\phi_{w,2,p}(0) - m\phi_{a,2,p}(0)\big) - \mathrm{i}\Omega\big(mF_{a,2}(0) - F_{w,2}(0)\big) + g(1-m)(\phi'_{a,2,p}(0) + Q_{a,2}) + \frac{\partial R_2}{\partial t}$$

$$\Lambda_3 = -\left.\frac{\partial \phi_{w,2,p}}{\partial z}\right|_{-h} \tag{S70}$$

SIMPLIFIED FORMS USED:

$$F_{w,2}(z=0) \simeq \sum \sigma^2[1-\sin^2\theta_a\delta_a^2(1-\cos^2(\varphi_2-\varphi)) + \frac{\delta_a^2 n^2}{2}]ZZ'\mathrm{e}^{i\Theta} \qquad \mathbf{O}(\sigma^2)+\mathbf{O}(\sigma^2\delta_a^2\sin^2\theta_a)$$

$$Q_{w,2}(z=0) \simeq -\mathrm{i}\sum s\sigma k 2\sin^2\theta_a\delta_a^2(1-\frac{1}{2}\cos^2(\varphi_2-\varphi))ZZ'\mathrm{e}^{i\Theta} \qquad \mathbf{O}(\sigma\delta_a^2\sin^2\theta_a)$$

$$F_{a,2}(z=0) \simeq \sum \sigma^2[1-\sin^2\theta_a\delta_a^2(1-\cos^2(\varphi_2-\varphi)) + \frac{3}{2}\delta_a^2]ZZ'\mathrm{e}^{i\Theta} \qquad \mathbf{O}(\sigma^2)+\mathbf{O}(\sigma^2\delta_a^2\sin^2\theta_a)$$

$$Q_{a,2}(z=0) \simeq -\mathrm{i}\sum s\sigma k 2\delta_a^2[1-\sin^2\theta_a(1-\frac{1}{2}\cos^2(\varphi_2-\varphi))]ZZ'\mathrm{e}^{i\Theta} \qquad \mathbf{O}(\sigma\delta_a^2)+\mathbf{O}(\sigma\delta_a^2\sin^2\theta_a)$$

$$\phi_{w,2,p}(z=0) \simeq \mathrm{i}\sum s\sigma\delta_a^2 n^2 ZZ'\mathrm{e}^{i\Theta} \qquad \mathbf{O}(\sigma\delta_a^2 n^2)$$

$$\left.\frac{\partial\phi_{w,2,p}}{\partial z}\right|_{z=0} \simeq \mathrm{i}\sum s\sigma 2k\delta_a^2 n^2 ZZ'\mathrm{e}^{i\Theta} \qquad \mathbf{O}(\sigma\delta_a^2 n^2)$$

$$\phi_{a,2,p}(z=0) \simeq \mathrm{i}\sum s\sigma\delta_a^2 ZZ'\mathrm{e}^{i\Theta} \qquad \mathbf{O}(\sigma\delta_a^2)+\mathbf{O}(\sigma\delta_a^2\sin^2\theta_a)$$

$$\left.\frac{\partial\phi_{a,2,p}}{\partial z}\right|_{z=0} \simeq -\mathrm{i}\sum s\sigma 2k\delta_a^2 ZZ'\mathrm{e}^{i\Theta} \qquad \mathbf{O}(\sigma\delta_a^2)$$

$$R_2 \simeq \sum -\sigma^2(1-m-\delta_a^2(3n^2/2+m))ZZ'\mathrm{e}^{i\Theta} \qquad \mathbf{O}(\sigma^2)$$

$$\frac{\partial R_2}{\partial t} \simeq \mathrm{i}s\sum 2\sigma^3(1-m-\delta_a^2(3n^2/2+m))ZZ'\mathrm{e}^{i\Theta} \qquad \mathbf{O}(\sigma^3)$$

$$\nu_\pm = 2\mathrm{i}\delta_a k\big(\pm\cos\theta_a - \mathrm{i}\frac{\delta_a}{4}\big)$$

$$\mu_\pm = 2\mathrm{i}\delta_a k\big(\mp\mathrm{i}l - \mathrm{i}\frac{\delta_a}{4}n^2\big)$$

$$\sin\theta_a = \frac{K\alpha_a}{\Omega},$$

$$\Omega \simeq 2\sigma, \qquad n = \frac{\alpha_a}{\alpha_w}, \qquad l = (\sin^2\theta_a - n^2)^{1/2},$$

$$\delta_w = \left(\frac{g}{k\alpha_a^2}\right)^{1/2}\frac{\alpha_a}{\alpha_w} = \delta_a n$$

### S4.1 Matrix 2x2 : BGKN73

When the ocean is assumed to have an infinite depth, we consider the atmosphere and ocean to be half spaces, with the continuity of velocity and pressure at $z = 0$ giving a 2 by 2 matrix,

$$M = \begin{pmatrix} \nu & -\mu_- \\ -m\Omega^2 - g(1-m)\nu & \Omega^2 \end{pmatrix} \tag{S71}$$

The solution is given by Cramer's method

$$A_+ = \frac{\det \mathbf{M}_1}{\det \mathbf{M}} \tag{S72}$$

with

$$\det \mathbf{M}_1 = \begin{vmatrix} \Lambda_1 & -\mu_- \\ \Lambda_2 & \Omega^2 \end{vmatrix} \tag{S73}$$

$$\det \mathbf{M} = \begin{vmatrix} \nu & -\mu_- \\ -m\Omega^2 - g(1-m)\nu & \Omega^2 \end{vmatrix} = \nu\Omega^2 - m\mu_-\Omega^2 - g(1-m)\nu\mu_- \tag{S74}$$

Here are the different pieces of $\det \mathbf{M}$,

- $\nu\Omega^2$ :

$$\begin{aligned} \nu\Omega^2 &= 4\sigma^2 \left( \frac{g}{2\alpha_a^2} + \mathrm{i}\frac{\Omega}{\alpha_a}\cos\theta_a \right) \\ &= \mathrm{i}8\sigma^2 k \left( -\mathrm{i}\frac{g}{4k\alpha^2} + \frac{\sigma}{k\alpha_a}\cos\theta_a \right) \\ &= 8\mathrm{i}\sigma^2 k\delta_a \left( \cos\theta_a - \mathrm{i}\frac{\delta_a}{4} \right) \end{aligned}$$

- $-m\mu_-\Omega^2$ :

$$\begin{aligned} -m\mu_-\Omega^2 &\simeq -m4\sigma^2 \cdot 2\mathrm{i}\delta_a k \left[ \mathrm{i}l - \mathrm{i}\frac{\delta_a}{2}n^2 \right] \\ &\simeq -8\mathrm{i}\sigma^2 k\delta_a m \left[ \mathrm{i}l \right] \end{aligned}$$

- $-g(1-m)\nu\mu_-$ :

$$\begin{aligned} -g(1-m)\nu\mu_- &\simeq -g(1-m)2\mathrm{i}\delta_a k \left( \cos\theta_a - \mathrm{i}\frac{\delta_a}{4} \right) 2\mathrm{i}\delta_a k \left( \mathrm{i}l + \mathrm{i}\frac{\delta_a}{4}n^2 \right) \\ &\simeq 4\sigma^2\delta_a k\delta_a (\mathrm{i}l\cos\theta_a + O(\delta_a)) \\ &\simeq 8\mathrm{i}\sigma^2 k\delta_a \left[ \frac{1}{2}\delta_a l \cos\theta_a \right] \end{aligned}$$

This gives $\det \mathbf{M}$, keeping only the second order in $\delta_a$ (the $\delta_a$ that is a factor should be remove alongside with all the factors in magenta when doing the ratio giving us a first order in $\delta_a$).

$$\det \mathbf{M} = \mathrm{i}8\sigma^2\delta_a k\left[-\mathrm{i}\frac{\delta_a}{4} + \cos\theta_a - \mathrm{i}ml + \frac{\delta_a}{2}\cos\theta_a l\right]$$

$$\det \mathbf{M} \simeq 8\mathrm{i}\sigma^2 k\delta_a\left[\cos\theta_a\left(1+\frac{\delta_a}{2}l\right) - \mathrm{i}\left(\frac{\delta_a}{4}+ml\right)\right] \quad \text{(S75)}$$

The term in $green$ is different from BGKN73 denominator. The difference is coming from the $\partial\zeta_2/\partial t$ term in the Bernoulli equation for the pressure at $z = 0$.

Now the numerator is,

$$\begin{aligned}
\det \mathbf{M}_1 &= \Lambda_1\Omega^2 + \mu_-\Lambda_2 \\
&= \Omega^2\left(\left.\frac{\partial\phi_{w,2,p}}{\partial z}\right|_0 - \left.\frac{\partial\phi_{a,2,p}}{\partial z}\right|_0 - Q_{a,2} + Q_{w,2}\right) + \mu_-\Big(-\Omega^2\big(\Phi_{w,2,p}(0) \\
&\quad -m\Phi_{a,2,p}(0)\big) - \mathrm{i}s\Omega\big(mF_{a,2}(0) - F_{w,2}(0)\big) + g(1-m)(\Phi'_{a,2,p}(0) + Q_{a,2}) + \frac{\partial R_2}{\partial t}\Big) \\
&\simeq \Omega^2\left(\left.\frac{\partial\phi_{w,2,p}}{\partial z}\right|_0 - \left.\frac{\partial\phi_{a,2,p}}{\partial z}\right|_0 - Q_{a,2} + Q_{w,2}\right) \\
&\quad +\mu_-\left(\mathrm{i}s\Omega F_{w,2}(0) + \frac{\partial R_2}{\partial t}\right) \qquad \text{(S76)}
\end{aligned}$$

where eqs. (S46), (S54) , (S62), (S56) give

$$\begin{aligned}
\Omega^2 \left.\frac{\partial \phi_{w,2,p}}{\partial z}\right|_0 &= 4\sigma^2 \cdot \mathrm{i} s\delta_a^2 n^2 \sigma 2k \\
&\simeq 8\mathrm{i}\sigma^2 k\delta_a s\sigma\delta_a n^2 && \text{(S77)}\\
-\Omega^2 \left.\frac{\partial \phi_{a,2,p}}{\partial z}\right|_0 &= -4\sigma^2 \cdot (-\mathrm{i}) s 2\sigma k\delta_a^2 \\
&\simeq 8\mathrm{i}\sigma^2 k\delta_a s\sigma\delta_a && \text{(S78)}\\
-\Omega^2 Q_{a,2} &\simeq -4\sigma^2 \cdot (-\mathrm{i}) s 2\sigma k\delta_a^2 \left(1-\sin^2\theta_a \left(1-\frac{1}{2}\cos^2(\varphi_2-\varphi)\right)\right) \\
&\simeq 8\mathrm{i}\sigma^2 k\delta_a \sigma s\delta_a \left(1-\sin^2\theta_a \left(1-\frac{1}{2}\cos^2(\varphi_2-\varphi)\right)\right) && \text{(S79)}\\
+\Omega^2 Q_{w,2} &\simeq 4\sigma^2 \cdot \mathrm{i}2\sigma sk\delta_a^2 \left(-\sin^2\theta_a \left(1-\frac{1}{2}\cos^2(\varphi_2-\varphi)\right)\right) \\
&\simeq 8\mathrm{i}\sigma^2 k\delta_a \sigma s\delta_a \left(-\sin^2\theta_a \left(1-\frac{1}{2}\cos^2(\varphi_2-\varphi)\right)\right) && \text{(S80)}\\
+\mathrm{i}s\Omega\mu_- F_{w,2}(0) &\simeq \mathrm{i}2s\sigma \cdot 2\delta_a k(\frac{\delta_a}{4}n^2 - l)\cdot \sigma^2[1-\sin^2\theta_a\delta_a^2(1-\cos^2(\varphi_2-\varphi)) + \delta_a^2 n^2/2] \\
&\simeq -8\mathrm{i}\sigma^2 k\delta_a \sigma s\left[\frac{l}{2} - \frac{\delta_a}{8}n^2 - \frac{l}{2}\sin^2\theta_a\delta_a^2\left[1-\cos^2(\varphi_2-\varphi)\right]\right] && \text{(S81)}\\
\mu_- \frac{\partial R_2}{\partial t} &= 2\delta_a k\left(\frac{\delta_a}{4}n^2 - l\right)\mathrm{i}s2\sigma^3(1-m) \\
&\simeq -8\mathrm{i}\delta_a k\sigma^2 \sigma s\left[\frac{l}{2} - \frac{\delta_a}{8}n^2\right]
\end{aligned}$$

Collecting all the terms we find,

$$\begin{aligned}
\det \mathbf{M}_1 &\simeq -8\mathrm{i}\sigma^2 k\delta_a\sigma s\left[l - \frac{\delta_a}{4}n^2 - \delta_a + 2\delta_a\sin^2\theta_a(1-\frac{1}{2}\cos^2\varphi_2) + \delta_a\left(n^2-1\right)\right] \\
&\simeq -8\mathrm{i}\sigma^2 k\delta_a\sigma s\left[l - \delta_a\left[2 - 2\sin^2\theta_a(1-\frac{1}{2}\cos^2\varphi_2) + \frac{5}{4}n^2\right]\right] \\
&\simeq -8\mathrm{i}\sigma^2 k\delta_a\sigma s\left[l - 2\delta_a\left[1 - \sin^2\theta_a(1-\frac{1}{2}\cos^2\varphi_2) + \frac{5}{8}n^2\right]\right] && \text{(S82)}
\end{aligned}$$

Then, we find the same expression as in BGKN73 numerator and the $\delta_a$ term is larger than the one in WG06, with 2 instead of 3/2.

The main term arises from the pressure boundary condition and from the difference between the pressure and the temporal derivative of the potential velocity.

We recall that the homogeneous atmospheric potential that radiates from the surface is given by eq. (S47),

$$\phi_{a,h,2}(z) = \sum sA_+ZZ'\mathrm{e}^{\nu_+z}\mathrm{e}^{\mathrm{i}\Theta}, \tag{S83}$$

with

$$\nu \simeq \frac{g}{2\alpha_a^2} + \mathrm{i}\frac{\Omega}{\alpha_a}\cos\theta_a \tag{S84}$$

and

$$A_+ \simeq -\sigma\frac{l - 2\delta_a\left[1 - \sin^2\theta_a\left(1 - \frac{1}{2}\cos^2(\varphi_2 - \varphi)\right) + \frac{5}{8}n^2\right]}{\cos\theta_a\left(1 + \frac{\delta_a}{2}l\right) - \mathrm{i}\left(\frac{\delta_a}{4} + ml\right)}. \tag{S85}$$

## S5 ADDING THE SOLID EARTH

The solid Earth is characterised by density $\rho_s$, compression velocity $\alpha_s$ and shear velocity $\beta$. Then the velocity potentials write as,

$$\begin{aligned} \phi_{w,2} &= \sum\left[\left(W_-\mathrm{e}^{\mu_-z} + W_+\mathrm{e}^{\mu_+z}\right)ZZ' + \Phi_{w,2,p}\right]\mathrm{e}^{\mathrm{i}\Theta}, \qquad \text{for } -h < z < \zeta \\ \phi_{a,2} &= \sum\left[sA_+\mathrm{e}^{\nu_+z}ZZ' + \Phi_{w,2,p}\right]\mathrm{e}^{\mathrm{i}\Theta}, \qquad \text{for } \zeta < z \end{aligned}$$

All the potentials share the same phase, $\Theta = \mathbf{K}\cdot\mathbf{x} - \Omega t$, $\Omega = s(\sigma + \sigma')$, but they differ by their vertical structures and amplitudes.

The boundary conditions for ocean/atmosphere interfaces remain the same. For the ocean bottom, the motion in the crust is given by velocity potentials for compression and shear waves in the solid Earth, we follow here the treatment in (Ardhuin & Herbers, 2013). Neglecting the effect of gravity, crustal motions can be separated into an irrotational part with a velocity potential $\phi_c$ and a rotational part with a stream function $\psi$, both solutions to Laplace's equation.

$$\phi_c = C_p\mathrm{e}^{\chi_p(z+h)}\mathrm{e}^{\mathrm{i}\Theta}, \tag{S86}$$

$$\psi = C_s\mathrm{e}^{\chi_s(z+h)}\mathrm{e}^{\mathrm{i}\Theta}, \tag{S87}$$

with

$$\chi_p = \sqrt{K^2 - \frac{\Omega^2}{\alpha_s^2}}, \quad \text{and} \quad \chi_s = \sqrt{K^2 - \frac{\Omega^2}{\beta^2}}. \tag{S88}$$

where $\alpha_s$ and $\beta$ are respectively the compression and the shear wave speed in the crust. Typically $\beta$ ranges from 2800 to 3200 m s$^{-1}$; $\alpha_s = \sqrt{3}\beta$. And $\rho_s \simeq 2500$ kg m$^{-3}$. The constants $C_p$ and $C_s$

have dimensions of $\mathrm{m}^2/\mathrm{s}$ and are determined by the boundary conditions at the ocean bottom. With $\lambda_e$ and $\mu_e$ the Lame elasticity parameters of the crust, Hooke's law of elasticity gives

$$\tau_{zz} = \lambda_e \left( \frac{\partial \xi_x}{\partial x} + \frac{\partial \xi_z}{\partial z} \right) + 2\mu_e \frac{\partial \xi_z}{\partial z}, \tag{S89}$$

$$\tau_{xz} = \mu_e \left( \frac{\partial \xi_x}{\partial z} + \frac{\partial \xi_z}{\partial x} \right). \tag{S90}$$

We recall that the compression and shear velocity are related to the Lame parameters,

$$\alpha_c^2 = \frac{\lambda_e + 2\mu_e}{\rho_s}, \tag{S91}$$

$$\beta^2 = \frac{\mu_e}{\rho_s}. \tag{S92}$$

The zero tangential stress on the ocean bottom $\tau_{xz}(z = -h) = 0$ yields the following relationship between $C_p$ and $C_s$, which is typical of seismic Rayleigh waves (Stoneley, 1926),

$$C_s = \frac{2\mathrm{i}K\chi_p}{\chi_s^2 + K^2} C_p = \frac{2\mathrm{i}\beta^2 K \chi_p}{2\beta^2 K^2 - \Omega^2} C_p. \tag{S93}$$

We can now eliminate $C_p$, using the continuity of the vertical velocity at the bottom,

$$\frac{\partial \phi_2}{\partial z} = \frac{\partial \phi_c}{\partial z} + \frac{\partial \psi}{\partial x} \quad \text{at} \quad z = -h \tag{S94}$$

$$W_+\mu_+ \mathrm{e}^{-\mu_+ h} + W_-\mu_- \mathrm{e}^{-\mu_- h} = \chi_p C_p + \mathrm{i}K C_s \tag{S95}$$

$$= \chi_p C_p + \mathrm{i}K \frac{2\mathrm{i}\beta^2 K \chi_p}{2\beta^2 K^2 - \Omega^2} C_p \tag{S96}$$

$$= \frac{\chi_p \Omega^2}{\Omega^2 - 2K^2\beta^2} C_p \tag{S97}$$

and the continuity of normal stresses, using the result from (Ardhuin et al., 2013) :

$$-\rho_w \frac{\partial \phi_2}{\partial t} = \tau_{zz} \quad z = -h \tag{S98}$$

$$\rho_w \Omega \mathrm{i} s \mathrm{e}^{-\mu_+ h} W_+ + \rho_w \Omega \mathrm{i} s W_- \mathrm{e}^{-\mu_- h} = r_{AH} C_p \tag{S99}$$

$$\tag{S100}$$

where

$$r_{AH} = \frac{\mathrm{i}s}{\Omega} \rho_s \left[ -\frac{4\beta^4 K^2 \chi_p \chi_s}{\Omega^2 - 2K^2\beta^2} + \left( \Omega^2 - 2K^2\beta^2 \right) \right]. \tag{S101}$$

Defining

$$r_\pm = \frac{\mathrm{i}s\rho_w\Omega\dfrac{\chi_p\Omega^2}{\Omega^2-2K^2\beta^2}}{\mu_\pm r_{AH}} \tag{S102}$$

$$= \frac{\mathrm{i}s\rho_w\Omega\dfrac{\chi_p\Omega^2}{\Omega^2-2K^2}}{\dfrac{\mathrm{i}}{\Omega}\mu_\pm\rho_s\left[-\dfrac{4\beta^4K^2\chi_p\chi_s}{\Omega^2-2K^2\beta^2}+\left(\Omega^2-2K^2\beta^2\right)\right]}, \tag{S103}$$

$$= \frac{\rho_w\chi_p\Omega^4}{\mu_\pm\rho_s\left[\left(\Omega^2-2K^2\beta^2\right)^2-4\beta^4K^2\chi_p\chi_s\right]} \tag{S104}$$

we combine these two boundary conditions by subtracting $r$ times the second equation to find a condition for the bottom velocities on the water side,

$$\mu^+(1-r_+)\mathrm{e}^{-\mu_+h}W_+ + \mu^-(1-r_-)\mu_-\mathrm{e}^{-\mu_-h}W_- = 0. \tag{S105}$$

We thus have the matrix equation

$$\mathbf{M}(A_+,\ W_-,\ W_+)^T = (\Lambda_1,\ \Lambda_2,\ 0)^T \tag{S106}$$

with

$$\mathbf{M} = \begin{pmatrix} \nu_+ & -\mu_- & -\mu_+ \\ -m\Omega^2-g(1-m)\nu_+ & \Omega^2 & \Omega^2 \\ 0 & (1-r_-)\,\mu_-\mathrm{e}^{-\mu_-h} & (1-r_+)\,\mu_+\mathrm{e}^{-\mu_+h} \end{pmatrix} \tag{S107}$$

and we use the following simplification,

$$\Lambda_1 = \left.\frac{\partial\Phi_{w,2,p}}{\partial z}\right|_0 - \left.\frac{\partial\Phi_{a,2,p}}{\partial z}\right|_0 - Q_{a,2}+Q_{w,2} \tag{S108}$$

$$\Lambda_2 = \mathrm{i}\Omega F_{w,2}(0)+\frac{\partial R_2}{\partial t} \tag{S109}$$

Assuming $mu_+ \simeq -\mu_- \simeq \mu$ the matrix equation simplifies as:

$$\mathbf{M} = \begin{pmatrix} \nu_+ & \mu & -\mu \\ -m\Omega^2-g(1-m)\nu_+ & \Omega^2 & \Omega^2 \\ 0 & -(1+r)\,\mu\mathrm{e}^{\mu h} & (1-r)\,\mu\mathrm{e}^{-\mu h} \end{pmatrix} \tag{S110}$$

# S6 FROM AMPLITUDE TO POWER

## S6.1 Particular case of a pair of wave trains

Here we first consider the pressure amplitude and variance in the water layer, which has been well studied and measured (Cox & Jacobs, 1989; Ardhuin et al., 2013).

In the case of only two wave trains of opposing direction with wave numbers $k_1$ and $k_2 \simeq -k_1$ with surface elevation

$$\zeta = a_1 \cos(k_1 x - \sigma_1 t) + a_2 \cos(k_2 x - \sigma_2 t) \tag{S111}$$

and velocity field

$$w(z=0) = a_1\sigma_1 \sin(k_1 x - \sigma_1 t) + a_2\sigma_2 \sin(k_2 x - \sigma_2 t) \tag{S112}$$

$$u(z=0) = a_1\sigma_1 \cos(k_1 x - \sigma_1 t) - a_2\sigma_2 \cos(k_2 x - \sigma_2 t) \tag{S113}$$

the second order pressure is, keeping only the small wavenumber components,

$$p_2 = \rho_w(u^2 + w^2) = -2\rho\sigma_1\sigma_2 a_1 a_2 \cos\left[Kx + \Omega t\right] \tag{S114}$$

Now we consider the variance of the pressure,

$$
\begin{aligned}
< p_2^2 > &= 4\rho_w^2\sigma_1^2\sigma_2^2 a_1^2 a_2^2/2 && \text{(S115)}\\
&= 2\rho_w^2 \sum_{k+k'=K} \sigma^2\sigma'^2 a^2 a'^2/2 && \text{(S116)}\\
&= 8\rho_w^2\sigma_1^2\sigma_2^2 \frac{a_1^2}{2}\frac{a_2^2}{2} && \text{(S117)}\\
&\simeq \frac{1}{2}\rho_w^2\Omega^4 E_1 E_2 && \text{(S118)}\\
&= \frac{1}{4}\rho_w^2\Omega^4 \sum_{k+k'=K} EE'. && \text{(S119)}
\end{aligned}
$$

## S6.2 Case of random waves

$$F_{p,2h}(\mathbf{K}, f_s) = 2 \lim_{|d\mathbf{K}|\to 0, df_s \to 0} \frac{\left\langle \left|P_{2h}^{+}\right|^2 \right\rangle}{dK_x dK_y df_s} \tag{S120}$$

with

$$
\begin{aligned}
P_{2h}^{s} &= \rho_a \mathcal{P}_{a,2,h} = -\rho_a \frac{\partial \phi_{a,2,h}}{\partial t} \\
&= -\rho_a \frac{\partial}{\partial t}\left( \frac{R_a(\mathbf{K})}{\rho_w 2\sigma'} p_{\text{surf}}^{s,s'}(\mathbf{K},\Omega) \right)
\end{aligned}
$$

remembering

$$p_{\text{surf}}^{s,s'}(\mathbf{K},\Omega) = \rho_w \sum_{\mathbf{k},s,\mathbf{k}',s'} D_z(\mathbf{k},s,\mathbf{k}',s') Z Z' \mathrm{e}^{\mathrm{i}\Theta} \quad \text{(S121)}$$

one gets :

$$
\begin{aligned}
P_{2h}^{s} &= \rho_a \sum_{\mathbf{k},s,\mathbf{k}',s'} \mathrm{i} R_a(\mathbf{K}) \frac{(s\sigma + s'\sigma')}{2\sigma'} D_z(\mathbf{k},s,\mathbf{k}',s') Z Z' \mathrm{e}^{\mathrm{i}\Theta} \\
&= \rho_a \sum_{\mathbf{k},s,\mathbf{k}'} \mathrm{i} s R_a(\mathbf{K}) \frac{(\sigma + \sigma')}{2\sigma'} D_z(\mathbf{k},s,\mathbf{k}',s) Z Z' \mathrm{e}^{\mathrm{i}\Theta} \quad \text{(S122)}
\end{aligned}
$$

Then,

$$
\begin{aligned}
2|P_{2h}^{+}|^2 &= 2\rho_a^2 \left| \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K},\sigma+\sigma'=\Omega} R_a(\mathbf{K}) \frac{(\sigma+\sigma')}{2\sigma'} D_z(\mathbf{k},+,\mathbf{k}',+) Z Z' \mathrm{e}^{\mathrm{i}\Theta} \right|^2 \\
&= 2\rho_a^2 \cdot 2 \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K},\sigma+\sigma'=\Omega} |R_a(\mathbf{K})|^2 \frac{(\sigma+\sigma')^2}{4\sigma'^2} |D_z(\mathbf{k},+,\mathbf{k}',+)|^2 |Z|^2 |Z'|^2 \quad \text{(S123)}
\end{aligned}
$$

And the spectrum density of the source writes :

$$F_{p,2h}(\mathbf{K},f_s) = \lim_{|d\mathbf{K}|\to 0, df_s \to 0} \frac{1}{K_x dK_y df_s} \sum_{\mathbf{k}+\mathbf{k}'=\mathbf{K},\sigma+\sigma'=\Omega} \frac{(\sigma+\sigma')^2}{\sigma'^2} R_a(\mathbf{K})^2 \rho_a^2 |D_z(\mathbf{k},+,\mathbf{k}',+)|^2 |Z|^2 |Z'|^2 \quad \text{(S124)}$$

using the definition :

$$E(k_x,k_y) = 2 \lim_{dk_x,dk_y \to 0} \frac{|Z|^2}{dk_x dk_y} \quad \text{(S125)}$$

$$F_{p,2h}(\mathbf{K},f_s) = \lim_{|d\mathbf{K}|\to 0, df_s \to 0} \frac{dk_x dk_x dk'_x dk'_y}{4 dK_x dK_y df_s} \sum_{\mathbf{k},s,\mathbf{k}'} \frac{(\sigma+\sigma')^2}{\sigma'^2} R_a(\mathbf{K})^2 \rho_a^2 |D_z(\mathbf{k},+,\mathbf{k}',+)|^2 E(k_x,k_y) E(k'_x,k'_y) \quad \text{(S126)}$$

Taking the limit to continuous sums and using a change of variable from $(k_x, k_y, k'_x, k'_y)$ to $(f_s, \varphi, K_x, K_y)$, with $K_x = k_x + k'_x$, $K_y = k_y + k'_y$ and $f_s = (\sqrt{gk} + \sqrt{gk'})/(2\pi)$ the Jacobian of the coordinate transform is

$$\det\left(\frac{\partial f_s \partial\varphi \partial K_x \partial K_y}{\partial k_x \partial k_y \partial k'_x \partial k'_y}\right) = \begin{vmatrix} g\cos\varphi/(4\pi\sigma) & -\sin\varphi/k & 1 & 0 \\ g\sin\varphi/(4\pi\sigma) & \cos\varphi/k & 0 & 1 \\ g\cos\varphi'/(4\pi\sigma') & 0 & 1 & 0 \\ g\sin\varphi'/(4\pi\sigma') & 0 & 0 & 1 \end{vmatrix} = \frac{g^2}{4\pi\sigma^3\sigma'}\left[\sigma' - \sigma\cos\left(\varphi - \varphi'\right)\right], \tag{S127}$$

$$\begin{aligned}\int F_{p,2h}(\mathbf{K}, f_s)\mathrm{d}K_x\mathrm{d}K_y\mathrm{d}f_s &= \rho_a^2 \int \frac{(\sigma+\sigma')^2}{4\sigma'^2}\left|R_a\right|^2\left|D_z\right|^2 E(k_x,k_y)E(k_x,k_y)\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k'_x\mathrm{d}k'_y \\ &= \rho_a^2 \int \frac{(\sigma+\sigma')^2}{4\sigma'^2}\left|R_a\right|^2\left|D_z\right|^2 \frac{E(k_x,k_y)E(k'_x,k'_y)4\pi\sigma^3\sigma'}{g^2\left[\sigma' - \sigma\cos\left(\varphi-\varphi'\right)\right]}\mathrm{d}f_s\mathrm{d}\varphi\mathrm{d}K_x\mathrm{d}K_y.\end{aligned}$$

To transform the spectra to frequency-direction spectra we use the Jacobian :

$$E(f,\varphi) = \frac{4\pi\sigma^3}{g^2}E(k_x,k_y) \tag{S128}$$

And then obtain :

$$\int F_{p,2h}(\mathbf{K}, f_s)\mathrm{d}K_x\mathrm{d}K_y\mathrm{d}f_s = \frac{1}{2}g^2\rho_a^2 \int f_s \frac{(\sigma+\sigma')}{4\sigma'^4}\left|R_a\right|^2\left|D_z\right|^2 \frac{E(f,\varphi)E(f',\varphi')}{\left[\sigma' - \sigma\cos\left(\varphi-\varphi'\right)\right]}\mathrm{d}f_s\mathrm{d}\varphi\mathrm{d}K_x\mathrm{d}K_y.$$

Now we use the unicity of the Fourier transform to identify the spectral density in the left and right hand sides and considering $|D_z(\mathbf{k},+,\mathbf{k}',+)| \simeq 2\sigma\sigma'$ :

$$F_{p,2h}(\mathbf{K}, f_s) = \frac{1}{2}g^2\rho_a^2 f_s \int_0^{2\pi} \frac{\sigma^2(\sigma+\sigma')}{\sigma'^2}\left|R_a\right|^2 \frac{E(f,\varphi)E(f',\varphi')}{\sigma' - \sigma\cos\left(\varphi-\varphi'\right)}\mathrm{d}\varphi. \tag{S129}$$

### S6.3 Acoustic energy in the water column

We take the acoustic energy per unit of horizontal surface to be twice the kinetic energy. Considering only $K < \Omega/\alpha_w$, we have

$$E_w = \rho_w \int_{-h}^{0} u^2 + w^2 \mathrm{d}z \tag{S130}$$

Now using eq. (45)

$$E_w = \rho_w \int_{-h}^{0} \sum\left(K^2 + \mu^2\right) W_-^2 \left(\frac{1+r}{1-r}\right)^2 \cos^2(|\mu| z)\mathrm{d}z \tag{S131}$$

$$= \rho_w \int_{\theta_{a,1}}^{\theta_{a,2}} \left(K^2 + |\mu|^2\right) F_{p,2h}(\theta_a, \varphi_2, f_s) \left|\frac{A}{P_{2,h}^+}\frac{W_-}{A}\frac{1+r}{1-r}\right|^2 \left(\frac{h}{2} + \frac{\sin 2|\mu| h}{4|\mu|}\right)\mathrm{d}\theta_a\mathrm{d}\varphi_2 \tag{S132}$$

with

$$\left|\frac{W_-}{A}\frac{1+r}{1-r}\right| = \left|\frac{2\nu(1+r)}{\mu\left[\mathrm{i}\sin(|\mu|h) + r\cos(|\mu|h)\right]}\right| \tag{S133}$$

and

$$\left|\frac{A}{P_{2,h}^{+}}\right| = \frac{1}{(\sigma+\sigma')\rho_a}. \tag{S134}$$

Now, looking at the ratio of the acoustic energy and radiated power for any $\theta_a$ and $\varphi_2$ we have,

$$Q_{max} = \frac{\Omega E_w}{F_{p,2h}(\theta_a, \varphi_2, f_s)/(\rho_a \alpha_a)} \tag{S135}$$

$$= \Omega \rho_w \rho_a \alpha_a \left(K^2 + |\mu|^2\right) \left|\frac{A}{P_{2,h}^{+}}\frac{W_-}{A}\frac{1+r}{1-r}\right|^2 \left(\frac{h}{2} + \frac{\sin 2|\mu|h}{4|\mu|}\right) \tag{S136}$$

$$= \frac{\rho_w \alpha_a}{\rho_a \Omega}\left(K^2 + |\mu|^2\right) \left|\frac{W_-}{A}\frac{1+r}{1-r}\right|^2 \left(\frac{h}{2} + \frac{\sin 2|\mu|h}{4|\mu|}\right) \tag{S137}$$

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